

Wednesday, February 5

Linear Versus Nonlinear Models

A *nonlinear* regression model is any model that *cannot* be written as

$$E(Y_i) = \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_k x_{ik},$$

such that $x_{i1}, x_{i2}, \dots, x_{ik}$ do not depend on any unknown parameters. A linear model must be *linear in the parameters*.

Example: Let's consider an exponential model for the `ToothGrowth` data, ignoring supplement type for now, such that

$$E(Y_i) = \beta_0 + \beta_1 2^{-d_i/h}$$

where Y_i is length and d_i is dose. If h is specified (say $h = 1$) we have a linear model that we can write as

$$E(Y_i) = \beta_0 + \beta_1 x_i,$$

where $x_i = 2^{-d_i/1}$. We can estimate this model in the usual way using `lm`.

```
m <- lm(len ~ I(2^(-dose/1)), data = ToothGrowth)
summary(m)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	35.14	1.555	22.60	1.942e-30
I(2^(-dose/1))	-33.61	2.988	-11.25	3.303e-16

But suppose we want to treat h as an *unknown parameter* to be *estimated* like β_0 and β_1 ? This would not be a linear model. We can write the model as

$$E(Y_i) = \beta_0 + \beta_1 x_i,$$

where $x_i = 2^{d_i/h}$, but now x_i depends on an unknown parameter (h) and so the model is not linear in the parameters and thus not a linear model.

Nonlinear Regression

The `nls` function can be used to estimate a *nonlinear* regression model (the `nls` stands for “nonlinear least squares”). But its arguments are a little different from `lm`.

1. The model must be written *mathematically* rather than *symbolically*. And this requires that we know the correct operators/functions in R corresponding to the desired mathematical operators/functions.
2. The *starting values* of the parameter estimates must be provided. This does two things: it identifies what parts of the model formula are parameters, and it provides initial values for an algorithm to use to solve the least squares optimization problem.

Example: First we will replicate the results for the *linear* model where h is known/specified, but now using `nls`.

```
m <- nls(len ~ beta0 + beta1*2^(-dose/1), data = ToothGrowth,
start = list(beta0 = 0, beta1 = 0))
summary(m)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t)
beta0	35.14	1.555	22.60	1.942e-30
beta1	-33.61	2.988	-11.25	3.303e-16

Note the starting values. For a *linear* model we (usually) do not need to provide good starting values so zeros work just fine. Now consider a *nonlinear* model where h is also an unknown parameter.

```
m <- nls(len ~ beta0 + beta1*2^(-dose/h), data = ToothGrowth,
  start = list(beta0 = 32, beta1 = -33, h = 1))
summary(m)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t)
beta0	27.9366	2.1482	13.005	1.062e-18
beta1	-36.6251	6.1143	-5.990	1.493e-07
h	0.4632	0.1459	3.174	2.422e-03

Specifying “good” starting values is important. What if we don’t provide good starting values?

```
m <- nls(len ~ beta0 + beta1*2^(-dose/h), data = ToothGrowth,
  start = list(beta0 = 0, beta1 = 0, h = 1))
```

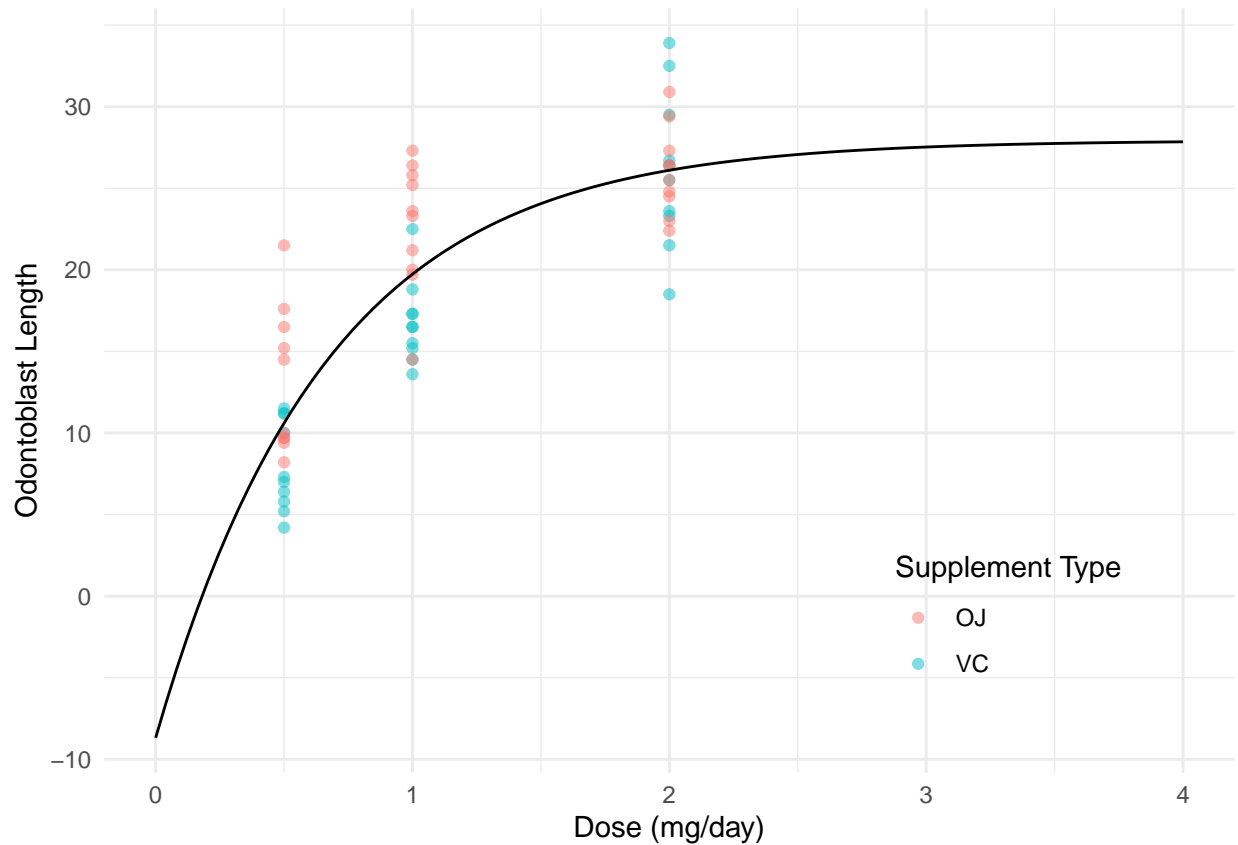
Error in nlsModel(formula, mf, start, wts, scaleOffset = scOff, nDcentral = nDcntr): singular gradient

How do we find good starting values? One approach is to use an approximate model like we did here that is linear. Another approach is to “eyeball” the estimates from a plot.

We can plot the model in the usual way.

```
d <- expand.grid(dose = seq(0, 4, length = 100))
d$yhat <- predict(m, newdata = d)

p <- ggplot(ToothGrowth, aes(x = dose, y = len)) +
  geom_point(aes(color = supp), alpha = 0.5) +
  geom_line(aes(y = yhat), data = d) +
  labs(x = "Dose (mg/day)", y = "Odontoblast Length",
  color = "Supplement Type") +
  theme_minimal() +
  theme(legend.position = "inside", legend.position.inside = c(0.8,0.2))
plot(p)
```

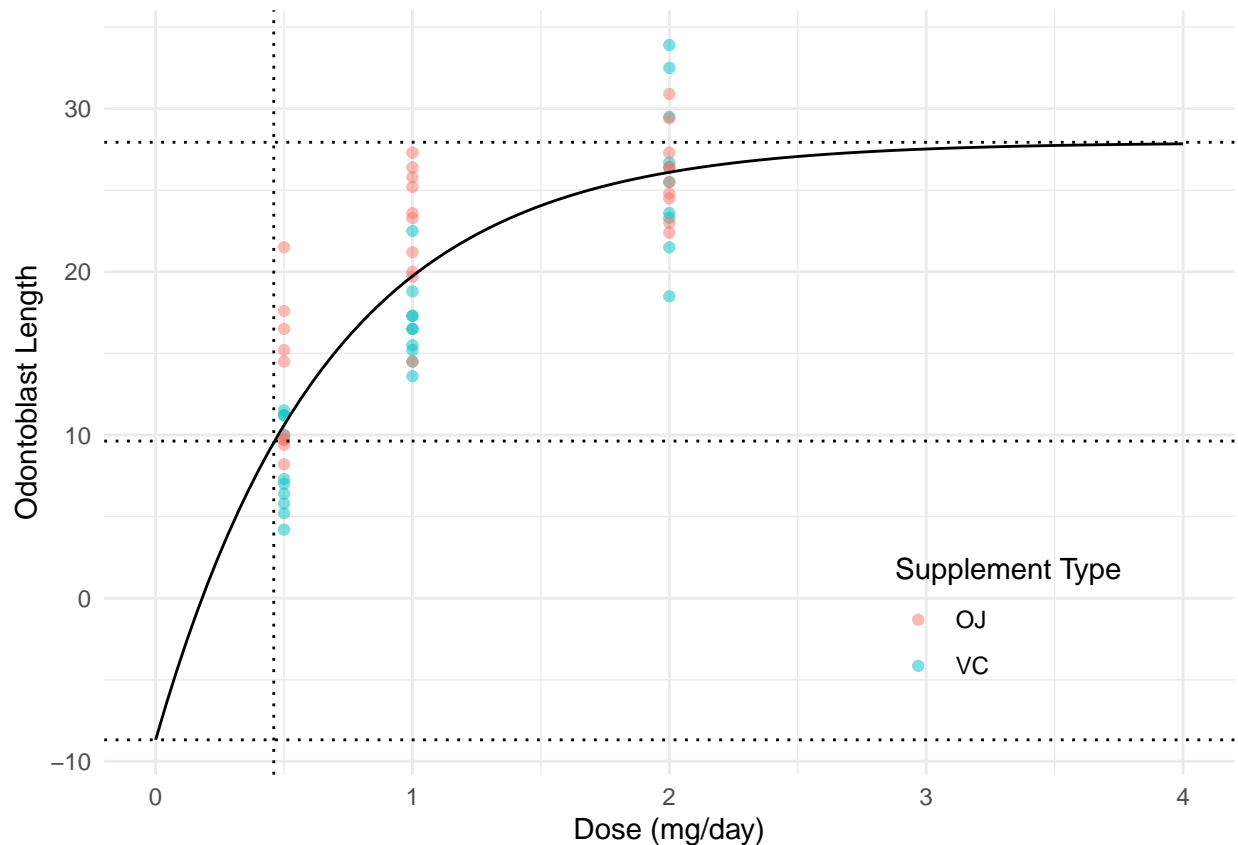


We can add some annotation if desired to highlight the interesting quantities.

```
summary(m)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t)
beta0	27.9366	2.1482	13.005	1.062e-18
beta1	-36.6251	6.1143	-5.990	1.493e-07
h	0.4632	0.1459	3.174	2.422e-03

```
p <- p + geom_hline(yintercept = 27.94, linetype = 3) # asymptote (b0)
p <- p + geom_hline(yintercept = 27.94 - 36.63, linetype = 3) # intercept (b0 + b1)
p <- p + geom_hline(yintercept = 27.94 - 36.63/2, linetype = 3) # half-way (b0 + b1/2)
p <- p + geom_vline(xintercept = 0.46, linetype = 3) # half-life (h)
plot(p)
```



Recall that the “intercept” is $\beta_0 + \beta_1$. We can make inferences concerning this quantity using `lincon`.

```
m <- nls(len ~ beta0 + beta1*2^(-dose/h), data = ToothGrowth,
  start = list(beta0 = 32, beta1 = -33, h = 0.75))
lincon(m, a = c(1,1,0)) # 1*b1 + 1*b2 + 0*h = b1 + b2
```

```
      estimate      se lower upper tvalue df  pvalue
(1,1,0),0  -8.688 7.562 -23.83  6.455 -1.149 57 0.2554
```

Does this make sense?

We can also replicate the estimates of the asymptote (β_0) and half-life (h) parameters using `lincon`.

```
cbind(summary(m)$coefficients, confint(m))
```

```
      Estimate Std. Error t value Pr(>|t|)      2.5%  97.5%
beta0  27.9366     2.1482  13.005 1.062e-18  24.7232  37.229
beta1 -36.6251     6.1143  -5.990 1.493e-07 -57.3146 -28.105
h       0.4632     0.1459   3.174 2.422e-03  0.2647   1.135
```

```
lincon(m, c(1,0,0)) # asymptote (beta0)
```

```
      estimate      se lower upper tvalue df  pvalue
(1,0,0),0   27.94  2.148  23.63  32.24     13 57 1.062e-18
```

```
lincon(m, c(0,0,1)) # half-life (h)
```

```
      estimate      se lower upper tvalue df  pvalue
(0,0,1),0   0.4632  0.1459  0.171  0.7554   3.174 57 0.002422
```

Note the difference in the confidence intervals (particularly for h). Here `confint` and `lincon` using different

kinds of confidence intervals: `confint` uses “profile-likelihood” intervals and `lincon` uses “Wald” intervals. We will discuss profile-likelihood confidence intervals later, but note here that typically they are more accurate.

The `emmeans` and `contrast` functions cannot (yet) be applied to a `nls` object. We must rely on something like `lincon` or clever parameterization (see below).

Now consider the model

$$E(Y_i) = \begin{cases} \beta_0 + \beta_1 2^{-x_i/h_{OJ}}, & \text{if the supplement type is OJ,} \\ \beta_0 + \beta_1 2^{-x_i/h_{VC}}, & \text{if the supplement type is VC,} \end{cases}$$

where x_i is dose. There are several ways we can handle case-wise models with `nls`: indicator variables, the `ifelse` function, and the `case_when` function.

1. We could write the model as

$$E(Y_i) = \beta_0 + \beta_1 2^{-x_i/(o_i h_{OJ} + v_i h_{VC})},$$

where o_i and v_i are indicator variables for the OJ and VC supplement types, respectively. In R we can program these indicator variables as `supp == "OJ"` and `supp == "VC"`, respectively. These will return `TRUE` or `FALSE`, but will be interpreted as 1 or 0, respectively, if used in a calculation. Here is how we can write this model in `nls`.

```
m <- nls(len ~ b0 + b1*2^(-dose/((supp == "OJ")*hoj + (supp == "VC")*hvc)),
  data = ToothGrowth, start = c(b0 = 28, b1 = -37, hoj = 0.46, hvc = 0.46))
summary(m)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t)
b0	27.5018	1.39516	19.712	7.258e-27
b1	-39.5856	5.47238	-7.234	1.422e-09
hoj	0.3382	0.06978	4.846	1.036e-05
hvc	0.5001	0.11208	4.462	3.963e-05

We could actually get away with one indicator variable if we are a little clever (and we are).

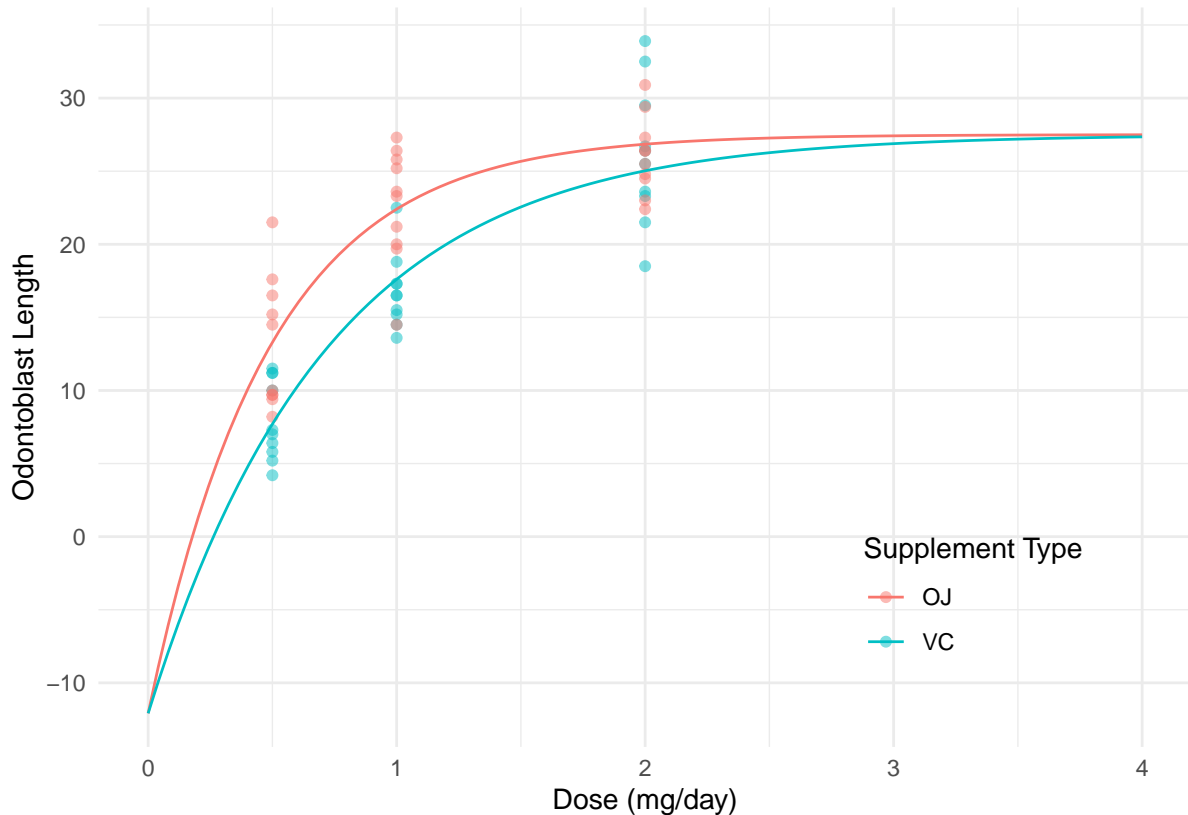
```
m <- nls(len ~ b0 + b1*2^(-dose/((supp == "OJ")*hoj + (1 - (supp == "OJ"))*hvc)),
  data = ToothGrowth, start = c(b0 = 28, b1 = -37, hoj = 0.46, hvc = 0.46))
summary(m)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t)
b0	27.5018	1.39516	19.712	7.258e-27
b1	-39.5856	5.47238	-7.234	1.422e-09
hoj	0.3382	0.06978	4.846	1.036e-05
hvc	0.5001	0.11208	4.462	3.963e-05

Here is a plot of the model with the data.

```
d <- expand.grid(dose = seq(0, 4, length = 100), supp = c("OJ","VC"))
d$yhat <- predict(m, newdata = d)

p <- ggplot(ToothGrowth, aes(x = dose, y = len, color = supp)) +
  geom_point(alpha = 0.5) +
  geom_line(aes(y = yhat), data = d) +
  labs(x = "Dose (mg/day)", y = "Odontoblast Length",
  color = "Supplement Type") +
  theme_minimal() +
  theme(legend.position = "inside", legend.position.inside = c(0.8,0.2))
plot(p)
```



2. When there are only two cases it can be convenient to use `ifelse`.

```
m <- nls(len ~ b0 + b1*2^(-dose/ifelse(supp == "OJ", hoj, hvc)),
  start = c(b0 = 28, b1 = -37, hoj = 0.46, hvc = 0.46),
  data = ToothGrowth)
summary(m)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t)
b0	27.5018	1.39516	19.712	7.258e-27
b1	-39.5856	5.47238	-7.234	1.422e-09
hoj	0.3382	0.06978	4.846	1.036e-05
hvc	0.5001	0.11208	4.462	3.963e-05

Here is another way we could write that using `ifelse`.

```
m <- nls(len ~ ifelse(supp == "OJ", b0 + b1*2^(-dose/hoj), b0 + b1*2^(-dose/hvc)),
  start = c(b0 = 28, b1 = -37, hoj = 0.46, hvc = 0.46),
  data = ToothGrowth)
summary(m)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t)
b0	27.5018	1.39516	19.712	7.258e-27
b1	-39.5856	5.47238	-7.234	1.422e-09
hoj	0.3382	0.06978	4.846	1.036e-05
hvc	0.5001	0.11208	4.462	3.963e-05

3. When there are more than two cases using `ifelse` can be tedious because we have to use nested `ifelse` functions. An easier approach is to use the `case_when` function from the **dplyr** package.

```
library(dplyr) # for case_when
m <- nls(len ~ case_when(
  supp == "OJ" ~ b0 + b1*2^(-dose/hoj),
  supp == "VC" ~ b0 + b1*2^(-dose/hvc),
), data = ToothGrowth, start = c(b0 = 28, b1 = -37, hoj = 0.46, hvc = 0.46))
summary(m)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t)
b0	27.5018	1.39516	19.712	7.258e-27
b1	-39.5856	5.47238	-7.234	1.422e-09
hoj	0.3382	0.06978	4.846	1.036e-05
hvc	0.5001	0.11208	4.462	3.963e-05

Ultimately it may be a matter of which is easiest to code.

Suppose we want to compare the two supplement types by making inferences about the *difference* $h_{VC} - h_{OJ}$?

```
summary(m)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t)
b0	27.5018	1.39516	19.712	7.258e-27
b1	-39.5856	5.47238	-7.234	1.422e-09
hoj	0.3382	0.06978	4.846	1.036e-05
hvc	0.5001	0.11208	4.462	3.963e-05

```
lincon(m, a = c(0,0,-1,1)) # 0*b0 + 0*b1 - 1*hoj + 1*hvc = hvc - hoj
```

	estimate	se	lower	upper	tvalue	df	pvalue
(0,0,-1,1),0	0.162	0.05532	0.05115	0.2728	2.928	56	0.004927

But suppose we parameterize the model as

$$E(Y_i) = \begin{cases} \beta_0 + \beta_1 2^{-x_i/h}, & \text{if the supplement type is OJ,} \\ \beta_0 + \beta_1 2^{-x_i/(h+\delta)}, & \text{if the supplement type is VC,} \end{cases}$$

so that h is the half-life parameter for OJ, $h + \delta$ is the half-life parameter for VC, and δ is the difference between them.

```
m <- nls(len ~ case_when(
  supp == "OJ" ~ b0 + b1*2^(-dose/h),
  supp == "VC" ~ b0 + b1*2^(-dose/(h + delta))
), data = ToothGrowth, start = c(b0 = 28, b1 = -37, h = 0.46, delta = 0))
summary(m)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t)
b0	27.5018	1.39516	19.712	7.258e-27
b1	-39.5856	5.47238	-7.234	1.422e-09
h	0.3382	0.06978	4.846	1.036e-05
delta	0.1620	0.05532	2.928	4.927e-03

Same model, different parameterization. Now we do not need to use `lincon` to obtain inferences for the difference between the two half life parameters, although we would need to use it to obtain inferences for the half life parameter for VC which is $h + \delta$.

```
lincon(m, a = c(0, 0, 1, 1))
```

	estimate	se	lower	upper	tvalue	df	pvalue
(0,0,1,1),0	0.5001	0.1121	0.2756	0.7247	4.462	56	3.963e-05

Interestingly using `confint` on these models “chokes” when trying to compute the profile-likelihood confidence interval for δ , although we can fix it by increasing the maximum number of iterations used in the estimation process (which is also used by `confint`).

```
confint(m)
```

```
Error in prof$getProfile(): number of iterations exceeded maximum of 50
```

```
m <- nls(len ~ b0 + b1*2^(-dose/iffelse(supp == "OJ", h, h + delta)),
  data = ToothGrowth, start = c(b0 = 28, b1 = -37, h = 0.46, delta = 0),
  control = nls.control(maxiter = 1000))
confint(m)
```

	2.5%	97.5%
b0	25.18143	30.8451
b1	-53.75676	-31.4127
h	0.23541	0.5268
delta	0.08001	0.2929

Alternatively we can easily compute a Wald confidence interval.

```
lincon(m, a = c(0,0,0,1)) # 0*b0 + 0*b1 + 0*h + 1*delta = delta
```

	estimate	se	lower	upper	tvalue	df	pvalue
(0,0,0,1),0	0.162	0.05532	0.05115	0.2728	2.928	56	0.004927

Actually if you omit the `a` argument `lincon` will return inferences for the model parameters like those given by `summary` and `confint` (but using Wald confidence intervals).

```
lincon(m)
```

	estimate	se	lower	upper	tvalue	df	pvalue
b0	27.5018	1.39516	24.70701	30.2967	19.712	56	7.258e-27
b1	-39.5856	5.47238	-50.54805	-28.6231	-7.234	56	1.422e-09
h	0.3382	0.06978	0.19838	0.4779	4.846	56	1.036e-05
delta	0.1620	0.05532	0.05115	0.2728	2.928	56	4.927e-03

This effectively causes `lincon` to use four sets of coefficients to make inferences about each parameter: `c(1,0,0,0)`, `c(0,1,0,0)`, `c(0,0,1,0)`, and `c(0,0,0,1)`.

Estimating a Linear Model with `nls`

We can use `nls` to estimate a linear model. Consider the following linear model.

```
m.whiteside <- lm(Gas ~ Insul + Temp + Insul:Temp, data = MASS::whiteside)
summary(m.whiteside)$coefficients
```

	Estimate	Std. Error	t value	Pr(> t)
(Intercept)	6.8538	0.13596	50.409	7.997e-46
InsulAfter	-2.1300	0.18009	-11.827	2.316e-16
Temp	-0.3932	0.02249	-17.487	1.976e-23
InsulAfter:Temp	0.1153	0.03211	3.591	7.307e-04

This can be written as

$$E(G_i) = \beta_0 + \beta_1 d_i + \beta_2 t_i + \beta_3 d_i t_i,$$

where G_i is gas consumption, t_i is temperature, and d_i is defined as

$$d_i = \begin{cases} 1, & \text{if the } i\text{-th observation is after insulation,} \\ 0, & \text{otherwise.} \end{cases}$$

Thus we can also write the model case-wise as

$$E(G_i) = \begin{cases} \beta_0 + \beta_2 t_i, & \text{if the } i\text{-th observation is before insulation,} \\ \beta_0 + \beta_1 + (\beta_2 + \beta_3) t_i, & \text{if the } i\text{-th observation is after insulation.} \end{cases}$$

Here are a few different ways to estimate this model using `nls`.

```
m1 <- nls(Gas ~ b0 + b1*(Insul == "After") + b2*Temp + b3*(Insul == "After")*Temp,
  data = MASS::whiteside, start = c(b0 = 0, b1 = 0, b2 = 0, b3 = 0))
summary(m1)$coefficients
```

```
Estimate Std. Error t value Pr(>|t|)
b0  6.8538    0.13596  50.409 7.997e-46
b1 -2.1300    0.18009 -11.827 2.316e-16
b2 -0.3932    0.02249 -17.487 1.976e-23
b3  0.1153    0.03211   3.591 7.307e-04
```

```
m2 <- nls(Gas ~ ifelse(Insul == "Before", b0 + b2*Temp,
  b0 + b1 + (b2 + b3)*Temp), data = MASS::whiteside,
  start = c(b0 = 0, b1 = 0, b2 = 0, b3 = 0))
summary(m2)$coefficients
```

```
Estimate Std. Error t value Pr(>|t|)
b0  6.8538    0.13596  50.409 7.997e-46
b1 -2.1300    0.18009 -11.827 2.316e-16
b2 -0.3932    0.02249 -17.487 1.976e-23
b3  0.1153    0.03211   3.591 7.307e-04
```

```
m3 <- nls(Gas ~ case_when(
  Insul == "Before" ~ b0 + b2*Temp,
  Insul == "After" ~ b0 + b1 + (b2 + b3)*Temp),
  data = MASS::whiteside,
  start = c(b0 = 0, b1 = 0, b2 = 0, b3 = 0))
summary(m3)$coefficients
```

```
Estimate Std. Error t value Pr(>|t|)
b0  6.8538    0.13596  50.409 7.997e-46
b1 -2.1300    0.18009 -11.827 2.316e-16
b2 -0.3932    0.02249 -17.487 1.976e-23
b3  0.1153    0.03211   3.591 7.307e-04
```

Estimated Expected Responses With `nls`

The `predict` function produces the estimated expected response for any combination of explanatory variables. For example,

```
d <- expand.grid(supp = c("VC", "OJ"), dose = c(0.5, 1, 1.5, 2))
d$yhat <- predict(m, newdata = d)
d
```

```
supp dose  yhat
1  VC  0.5  7.705
2  OJ  0.5 13.297
3  VC  1.0 17.602
4  OJ  1.0 22.405
5  VC  1.5 22.551
6  OJ  1.5 25.673
7  VC  2.0 25.026
```

```
8 OJ 2.0 26.845
```

However `predict` does not provide standard errors or confidence intervals for estimating expected responses based on `nls`, and we cannot use `contrast` with `nls`. But we can use the function `nlsint` from the `trtools` package to get approximate standard errors and confidence or prediction intervals from a `nls` object.

```
library(trtools)
nlsint(m, newdata = d)
```

	fit	se	lwr	upr
1	7.705	1.0885	5.525	9.886
2	13.297	0.9857	11.322	15.272
3	17.602	0.9868	15.625	19.579
4	22.405	0.7711	20.860	23.949
5	22.551	0.8758	20.796	24.305
6	25.673	0.7342	24.202	27.144
7	25.026	0.7322	23.559	26.493
8	26.845	1.0098	24.823	28.868

```
cbind(d, nlsint(m, newdata = d))
```

	supp	dose	yhat	fit	se	lwr	upr
1	VC	0.5	7.705	7.705	1.0885	5.525	9.886
2	OJ	0.5	13.297	13.297	0.9857	11.322	15.272
3	VC	1.0	17.602	17.602	0.9868	15.625	19.579
4	OJ	1.0	22.405	22.405	0.7711	20.860	23.949
5	VC	1.5	22.551	22.551	0.8758	20.796	24.305
6	OJ	1.5	25.673	25.673	0.7342	24.202	27.144
7	VC	2.0	25.026	25.026	0.7322	23.559	26.493
8	OJ	2.0	26.845	26.845	1.0098	24.823	28.868

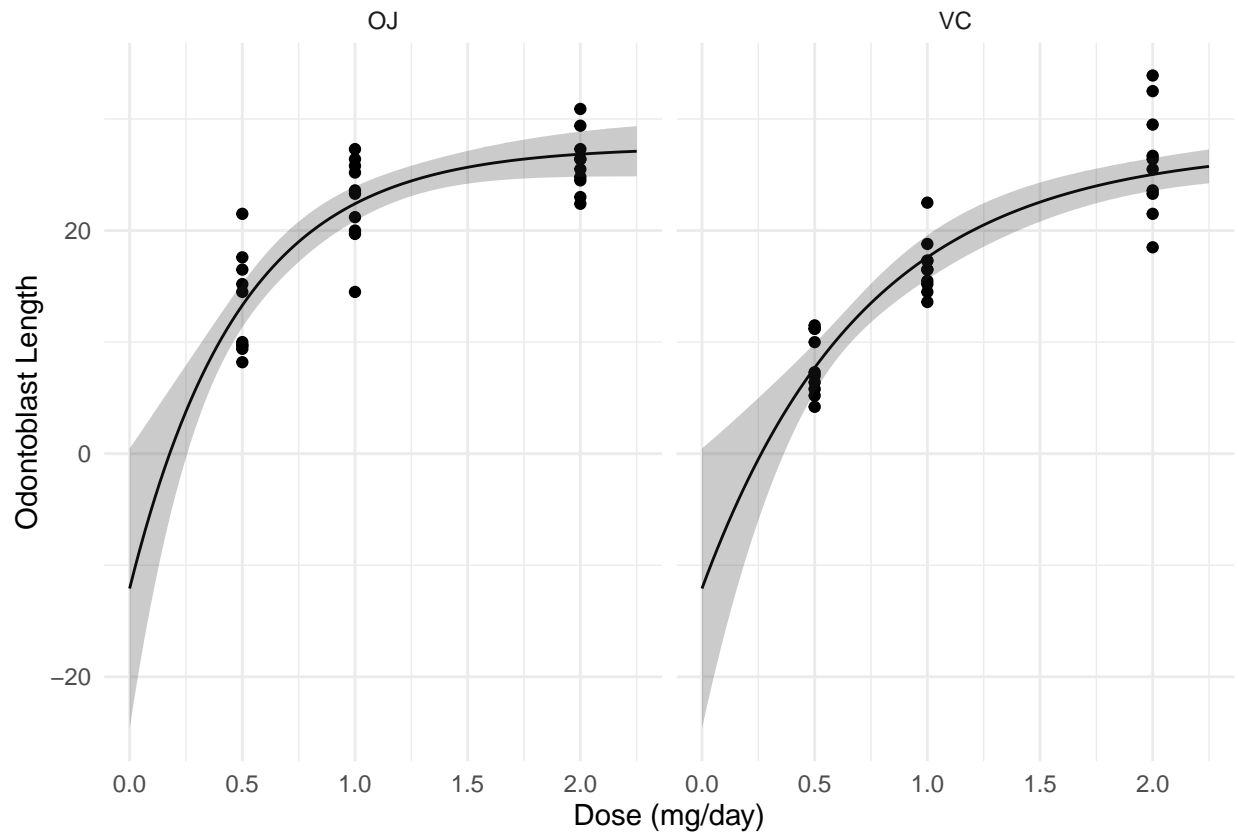
The intervals are confidence intervals by default. Prediction intervals can be obtained using `interval = prediction` (as you would with the `predict` function with a linear model).

The reason why `predict` does not provide standard errors and thus confidence intervals for a `nls` object is that the estimated expected response is *not* a linear function of the model parameters in a nonlinear model. The `nlsint` function uses what is called the *delta method* to come up with an approximate standard error. We will discuss the delta method later.

The `nlsint` function is also useful for plotting confidence and/or prediction intervals.

```
d <- expand.grid(supp = c("OJ", "VC"), dose = seq(0, 2.25, length = 100))
d <- cbind(d, nlsint(m, newdata = d))

p <- ggplot(ToothGrowth, aes(x = dose, y = len)) +
  geom_point() + facet_wrap(~ supp) + theme_minimal() +
  geom_line(aes(y = fit), data = d) +
  geom_ribbon(aes(x = dose, ymin = lwr, ymax = upr, y = NULL),
    alpha = 0.25, data = d) +
  labs(x = "Dose (mg/day)", y = "Odontoblast Length")
plot(p)
```



```
d <- expand.grid(supp = c("OJ", "VC"), dose = seq(0, 2.25, length = 100))
d <- cbind(d, nlsint(m, newdata = d, interval = "prediction"))
p <- p + geom_ribbon(aes(x = dose, ymin = lwr, ymax = upr, y = NULL),
  alpha = 0.25, data = d)
plot(p)
```

