Monday, October 6

Bias and Mean Squared Error of Ratio Estimators

Ratio estimators are (usually) *biased*.

1. What does it mean to say that a ratio estimator is biased?
2. When is the bias small for ratio estimators?
3. Why do we usually tolerate bias in ratio estimators?

Suppose that θ is a parameter and $\hat{\theta}$ is an estimator. An expression of the "imprecision" of $\hat{\theta}$ for estimating θ is the **mean squared error** (MSE) defined as

$$E[(\hat{\theta} - \theta)^2] = \sum_{j=1}^{J} (\hat{\theta}_j - \theta)^2 P(\mathcal{S}_j),$$

for a sample space of J samples, where the j-th sample is denoted as $\mathcal{S}_j$, and the estimate produced by that sample is denoted as $\hat{\theta}_j$.

The mean squared error can be usefully decomposed into two terms as

$$E[(\hat{\theta} - \theta)^2] = [E(\hat{\theta}) - \theta]^2 + V(\hat{\theta}),$$

where $E(\hat{\theta}) - \theta$ is the *bias* of the estimator, and $V(\hat{\theta})$ is the *variance* of the estimator. Sometimes we can increase one while decreasing the other leading to a *bias-variance trade-off*.

Double Sampling for Ratio Estimators

Under simple random sampling the ratio estimators of μ_y and τ_y are

$$\hat{\mu}_y = \frac{\bar{y}}{\bar{x}}\mu_x \quad \text{and} \quad \hat{\tau}_y = \frac{\bar{y}}{\bar{x}}\tau_x,$$

respectively. But we cannot use these estimators of the corresponding parameter for the auxiliary variable (i.e., μ_x or τ_x) is *unknown*. One solution is to use **double sampling**.

1. Using simple random sampling, obtain a sample of size n' and compute $\hat{\mu}_x = \bar{x}'$ or $\hat{\tau}_x = N\bar{x}'$, where $\bar{x}'$ is the mean of the auxiliary variable for the n' sampled elements.
2. Select a sample of size $n < n'$ from the first sample using simple random sampling and compute the ratio estimator
$$\hat{\mu}_y = \frac{\bar{y}}{\bar{x}}\hat{\mu}_x \quad \text{or} \quad \hat{\tau}_y = \frac{\bar{y}}{\bar{x}}\hat{\tau}_x.$$

Example: Recall our example of the population of 744 leaves from a shining gum (*Eucalyptus nitens*). We had two measurements of leaf area: a precise calculation and a crude approximation using the produce of leaf length and width. But suppose that we had not measured the crude area approximation for all 744 leaves. Instead suppose we had used a double sampling design where the first sample was $n' = 20$ leaves and the second sample was $n = 10$ leaves.

i	Area	Length × Width
1	80.7	113.08
2	69.7	98.40
3	66.1	97.17
4		198.40
5		103.20
6		55.10
7		55.10
8	31.5	43.96
9		28.35
10		61.10
11	28.5	39.15
12		55.35
13	73.4	101.43
14	48.4	58.80
15	74.1	102.50
16		36.57
17	16.7	26.28
18		30.20
19		143.63
20	29.4	40.00

The mean crude area approximation of the 20 leaves in the first sample is 74.39 square cm, the mean area of the 10 leaves in the second sample is 51.85 square cm, and the mean crude area approximation for these 10 leaves is 72.08 square cm. What is are the estimates of μ_y and τ_y using a ratio estimator?

Optimum Allocation for Double Sampling for Ratio Estimators

How should we decide on the two sample sizes: n' and n?

The estimated variance of $\hat{\mu}_y$ here is

$$V(\hat{\mu}_y) \approx \underbrace{\left(1 - \frac{n'}{N}\right)\frac{\sigma^2}{n'}}_{a} + \underbrace{\left(1 - \frac{n}{n'}\right)\frac{\sigma_r^2}{n}}_{b},$$

where

$$\sigma^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - \mu_y)^2 \quad \text{and} \quad \sigma_r^2 = \frac{1}{N-1}\sum_{i=1}^{N}(y_i - Rx_i)^2.$$

Term a is due to first phase sampling, and term b is due to the second phase sampling.

Assume a total cost of the survey of $C = c_x n' + c_y n$, where

$$c_x = \text{cost per unit for observing } x_i,$$
$$c_y = \text{cost per unit for observing } y_i.$$

Note that we only need to observe the auxiliary variable (x_i) in the first phase, and the target variable (y_i) is only observed in the second phase. For a fixed total cost of C, the variance of $\hat{\mu}_y$ and $\hat{\tau}_y$ is minimized if

$$\frac{n}{n'} = \sqrt{\frac{c_x}{c_y}\left(\frac{1}{\sigma^2/\sigma_r^2 - 1}\right)},$$

assuming $\sigma^2 > \sigma_r^2$. The optimum allocation is then

$$n' = C/(c_x + fc_y) \quad \text{and} \quad n = fC/(c_x + fc_y),$$

where $f = n/n'$.

Example: Suppose that the cost of observing the auxiliary variable is $c_x = 0.5$, the cost of observing the target variable is $c_y = 1$, and the relative efficiency is $\sigma^2/\sigma_r^2 = 4$. For a fixed total cost of $C = 100$, we can show that $f \approx 0.41$, $n' \approx 110$, and $n \approx 45$.

Note that we must have $n' > n$ if we are to use double sampling. There are two limiting cases to consider.

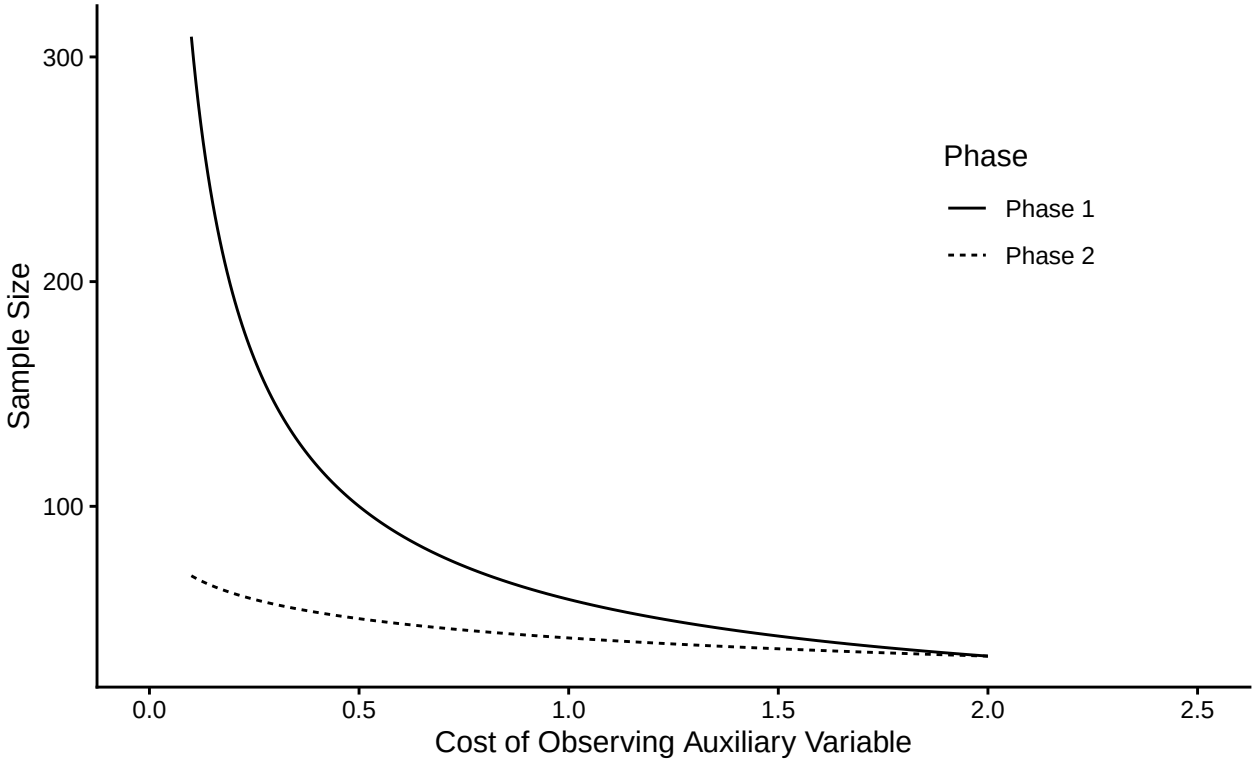
1. If $n' = n$ then double sampling reduces to a simple random sampling design where we just use the first phase sample and the estimators become $\hat{\mu}_y = \bar{y}$ and $\hat{\tau}_y = N\bar{y}$ (i.e., we are not using the ratio estimator).
2. If $n' = N$ then double sampling reduces to a simple random sampling design where the first phase sample is a *census* and the estimators become the usual ratio estimators

   $$\hat{\mu}_y = \frac{\bar{y}}{\bar{x}}\mu_x \quad \text{and} \quad \hat{\tau}_y = \frac{\bar{y}}{\bar{x}}\tau_x,$$

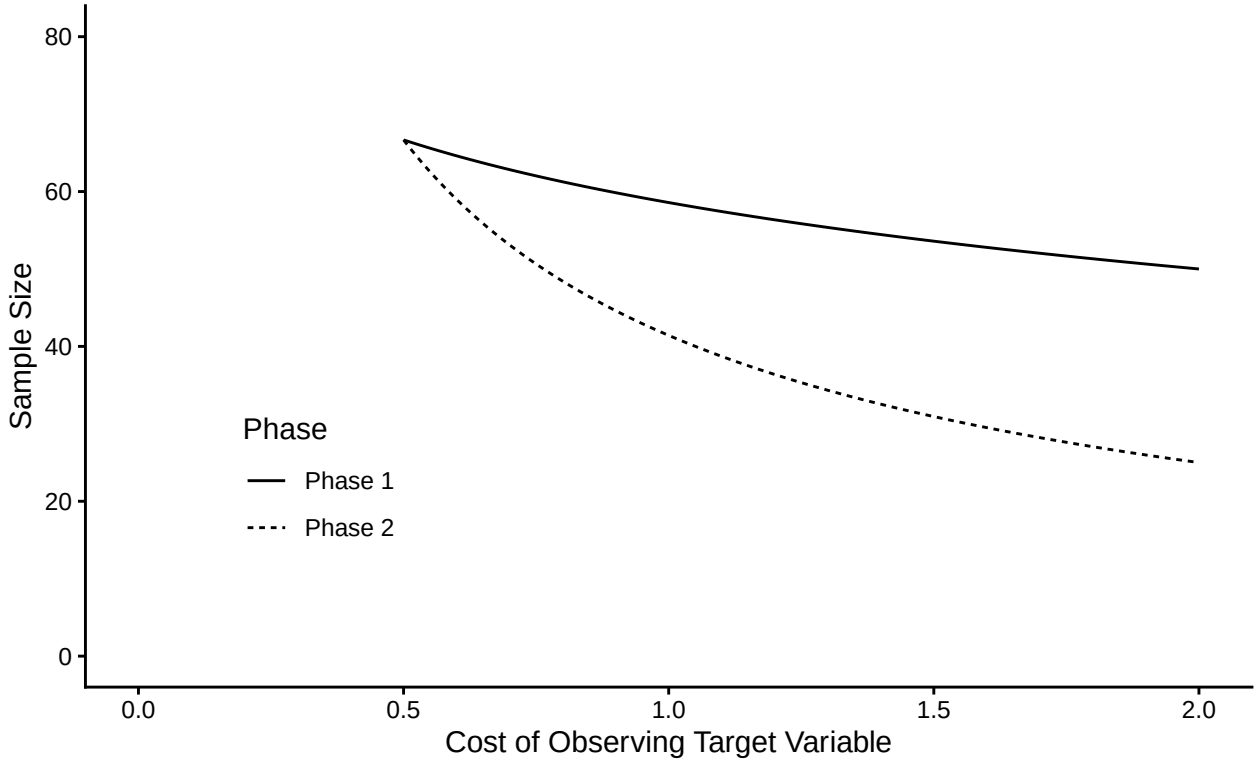
   as μ_x and τ_x will be *known* and need not be estimated.

When is using double sampling with a ratio estimator better than using simple random sampling without a ratio estimator?

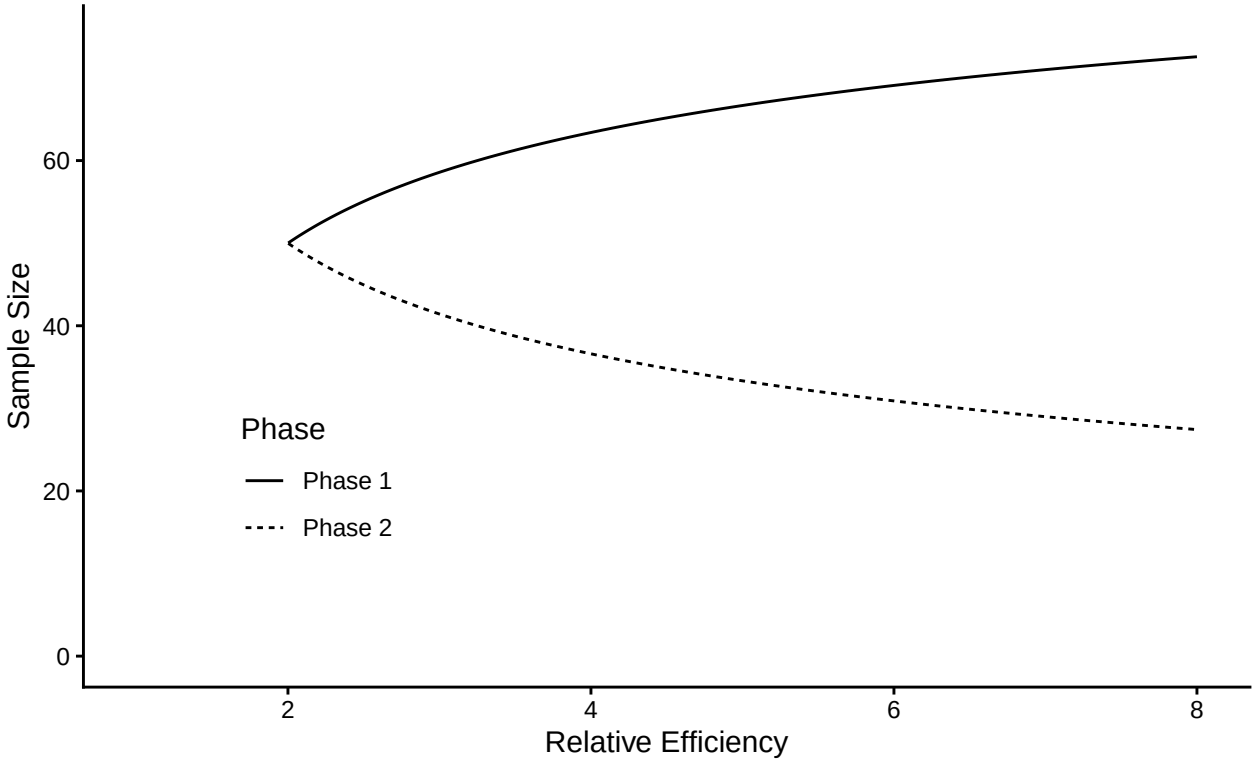
Example: Suppose that $C = 100$, $c_y = 1$, and $\sigma^2/\sigma_r^2 = 3$. What happens as we increase c_x (i.e., the cost of observing the auxiliary variable)?



Example: Suppose that $C = 100$, $c_x = 1$, and $\sigma^2/\sigma_r^2 = 3$. What happens as we increase c_y (i.e., the cost of observing the target variable)?



Example: Suppose that $C = 100$, $c_y = 1$, and $c_x = 1$. What happens as we increase σ^2/σ_r^2 (i.e., the relative efficiency of the ratio estimator)?



Ratio Estimators Under Stratified Random Sampling

There are a couple of different ways to use a ratio estimator with stratified random sampling.

The Separate Ratio Estimator

Estimate the mean or total of each stratum *separately* using a ratio estimator.

1. Estimate $\mu_{y,j}$ or $\tau_{y,j}$ *separately* for each stratum using a ratio estimator so that

$$\hat{\mu}_{y,j} = \frac{\bar{y}_j}{\bar{x}_j}\mu_{x,j} \quad \text{or} \quad \hat{\tau}_{y,j} = \frac{\bar{y}_j}{\bar{x}_j}\tau_{x,j},$$

   respectively.

2. Estimate μ_y or τ_y using

$$\hat{\mu}_y = \sum_{j=1}^{L} \frac{N_j}{N}\hat{\mu}_{y,j} \quad \text{and} \quad \hat{\tau}_y = \sum_{j=1}^{L} \hat{\tau}_{y,j},$$

   respectively.

The Combined Ratio Estimator

Use one ratio estimator based on sample means that *combine* the means from the strata.

1. Compute

$$\bar{y}_{st} = \sum_{j=1}^{L} \frac{N_i}{N}\bar{y}_j \quad \text{and} \quad \bar{x}_{st} = \sum_{j=1}^{L} \frac{N_i}{N}\bar{x}_j.$$

2. Estimate μ_y or τ_y using the ratio estimators

$$\hat{\mu}_y = \frac{\bar{y}_{st}}{\bar{x}_{st}}\mu_x \quad \text{or} \quad \hat{\tau}_y = \frac{\bar{y}_{st}}{\bar{x}_{st}}\tau_x,$$

   respectively.

Example: Consider the following summary from a stratified random sampling design.

Stratum	N_j	n_j	$\bar{y}_j$	$\bar{x}_j$	$\mu_{x,j}$
1	500	100	90	50	95
2	400	50	80	45	78
3	100	25	70	40	72

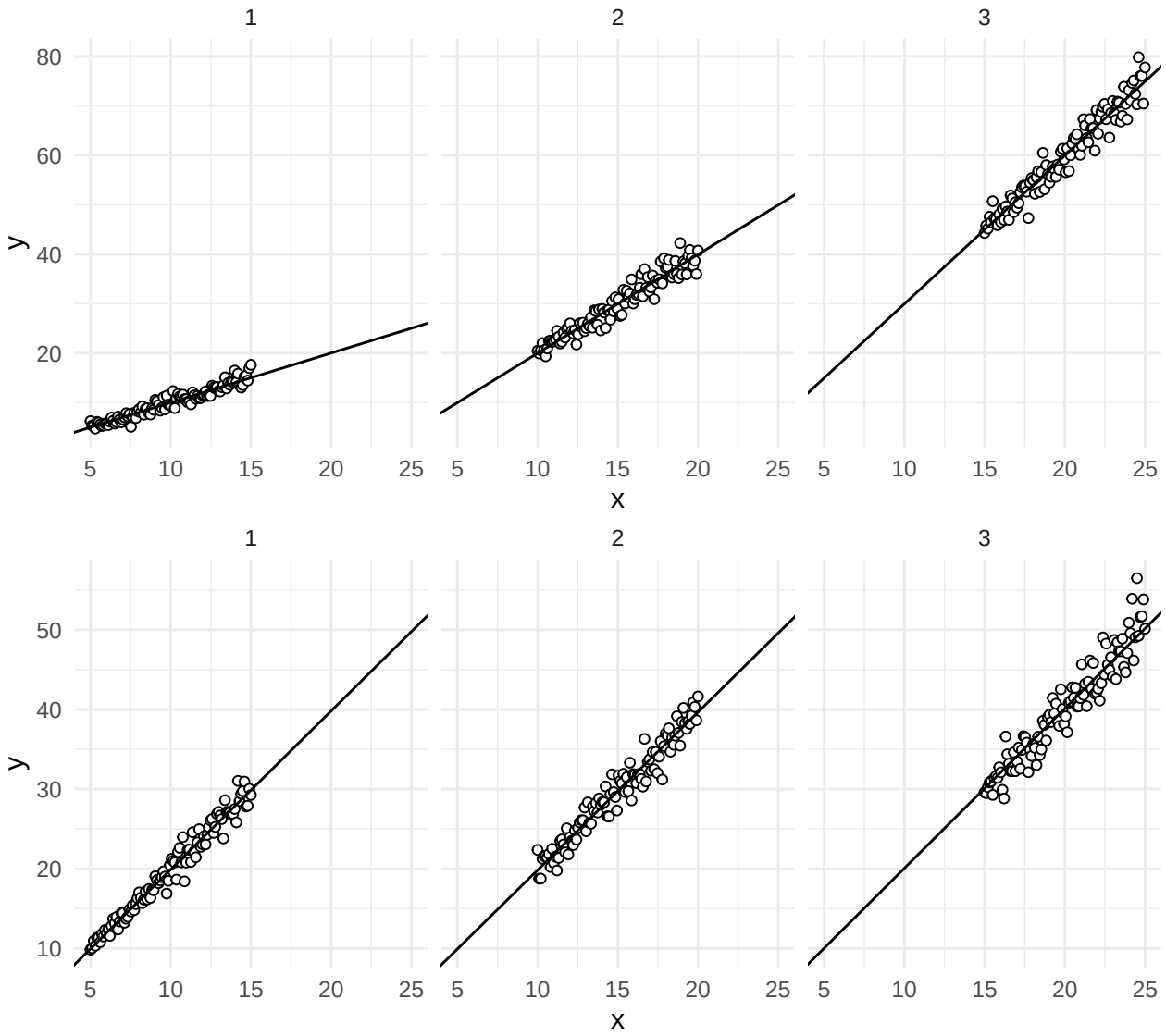
Also $\mu_x = 85.9$. The *separate* ratio estimator of μ_y is

$$\hat{\mu}_y = \frac{500}{1000}\underbrace{\left[\left(\frac{90}{50}\right)95\right]}_{\hat{\mu}_{y,1}} + \frac{400}{1000}\underbrace{\left[\left(\frac{80}{45}\right)78\right]}_{\hat{\mu}_{y,2}} + \frac{100}{1000}\underbrace{\left[\left(\frac{70}{40}\right)72\right]}_{\hat{\mu}_{y,3}}.$$

The *combined* ratio estimator is

$$\hat{\mu}_y = \frac{\overbrace{\left(\frac{500}{1000}\right)90 + \left(\frac{400}{1000}\right)80 + \left(\frac{100}{1000}\right)70}^{\bar{y}_{st}}}{\underbrace{\left(\frac{500}{1000}\right)50 + \left(\frac{400}{1000}\right)45 + \left(\frac{100}{1000}\right)40}_{\bar{x}_{st}}}85.9.$$

Which one should we use?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