Wednesday, Oct 2

Regression Estimator

Assume a simple random sampling design. The *regression estimator* of μ_y is

$$\hat{\mu}_y = \bar{y} + b(\mu_x - \bar{x}).$$

The regression estimator of τ_y is N times this estimator:

$$\hat{\tau}_y = N\bar{y} + b(\tau_x - N\bar{x}).$$

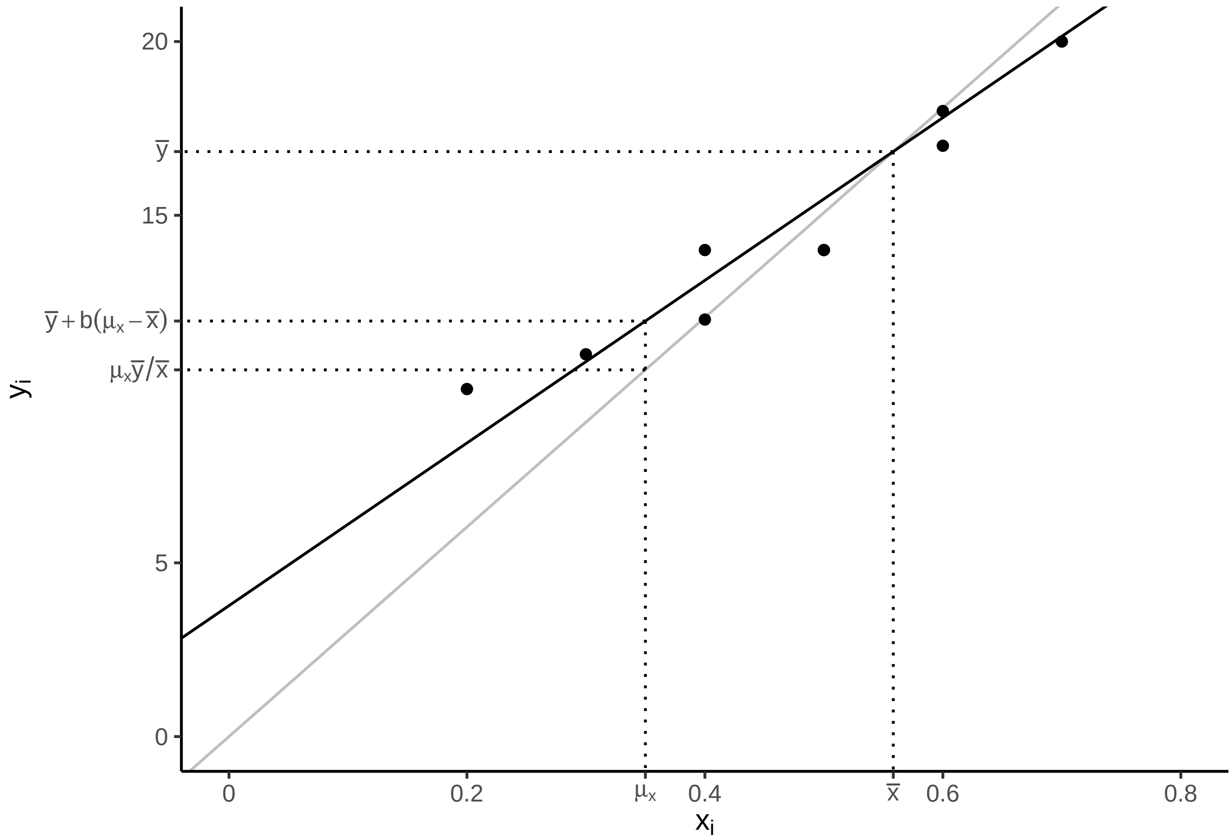
In both estimators $b = \hat{\rho}s_y/s_x$, where $\hat{\rho}$ is the correlation between the target and auxiliary variable for the elements in the sample, s_y is the standard deviation of the target variable for the elements in the sample, and s_x is the standard deviation of the auxiliary variable for the elements in the sample.

The regression estimator for μ_y can also be written as

$$\hat{\mu}_y = \underbrace{\bar{y} - b\bar{x}}_{a} + b\mu_x = a + b\mu_x,$$

where $a = \bar{y} - b\bar{x}$ and $b = \hat{\rho}s_y/s_x$ are the intercept and slope, respectively, of a *regression line*. Contrast this with the *ratio estimator* for μ_y which can be written as $\hat{\mu}_y = r\mu_x$, which is based on a different regression line that has an intercept of zero.[1]

[1] The slope and intercept for the regression estimator can be found using standard software for ordinary least squares regression. But note that the slope of a line corresponding to a ratio estimator is $\bar{y}/\bar{x}$. This is equivalent to the slope of a regression line *with no intercept* estimated using *weighted least squares* where the weights are the reciprocals of the auxiliary variable values. In R this would be something like `lm(y ~ -1 + x, weights = 1/x, data = mydata)`.



Estimated Variance of a Regression Estimator

Assuming simple random sampling, the estimated variance of $\hat{\mu}_y = \bar{y} + b(\mu_x - \bar{x})$ can be written as

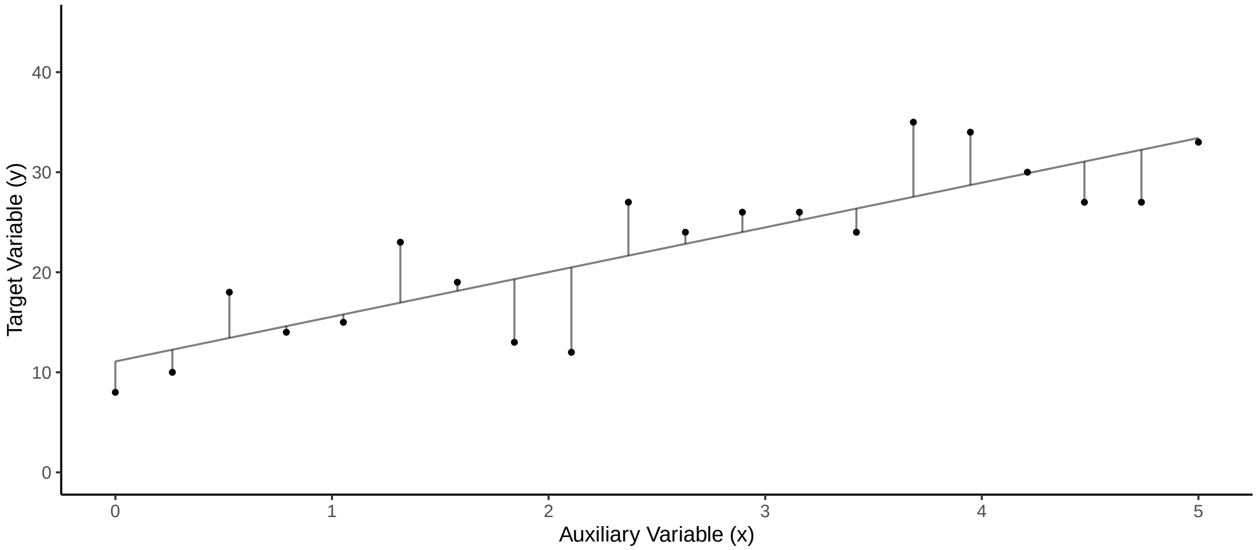
$$\hat{V}(\hat{\mu}_y) = \left(1 - \frac{n}{N}\right) \frac{\sum_{i \in \mathcal{S}} (y_i - a - bx_i)^2/(n-2)}{n}.$$

The term $\sum_{i \in \mathcal{S}} (y_i - a - bx_i)^2$ can also be computed as $\sum_{i \in \mathcal{S}} (y_i - a - bx_i)^2 = (n-1)s_y^2(1-\hat{\rho}^2)$, which shows that it gets smaller as the correlation gets larger (in absolute value).

The estimated variance of $\hat{\tau}_y = N\bar{y} + b(\tau_x - N\bar{x})$ is then

$$\hat{V}(\hat{\tau}_y) = N^2 \left(1 - \frac{n}{N}\right) \frac{\sum_{i \in \mathcal{S}} (y_i - a - bx_i)^2/(n-2)}{n}.$$

Example: The figure below show the values of the target and auxiliary variable for a sample of 20 from a population of size 100.



We have the following summary statistics: $\bar{y} = 22.2$, $\bar{x} = 2.5$, $\mu_x = 3$, $s_y = 8.2$, $s_x = 1.6$, and $\hat{\rho} = 0.8$. The line shown above has intercept $a = \bar{y} - b\bar{x}$ and slope $b = \hat{\rho} s_y / s_x$.

1. Assuming a simple random sampling design, what is the estimate and the bound on the error of estimation for estimating μ_y using the *regression estimator* if $\sum_{i \in \mathcal{S}} (y_i - a - bx_i)^2 = 353.3$.

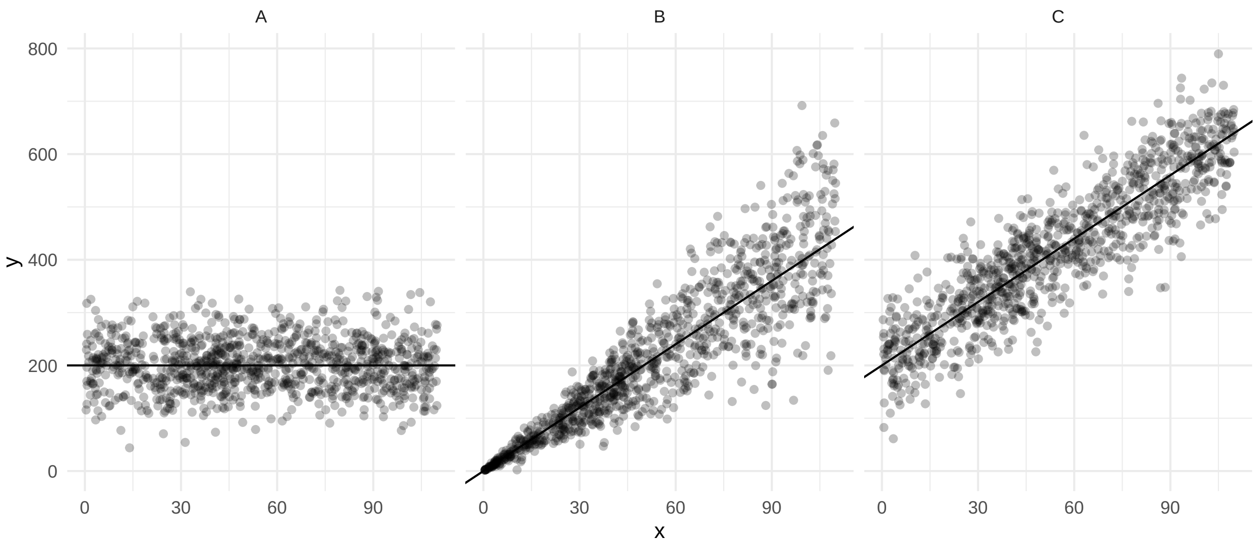
2. Assuming a simple random sampling design, what is the estimate and bound on the error of estimation using the *ratio estimator* if $\sum_{i \in \mathcal{S}} (y_i - rx_i)^2 = 1258.9$.

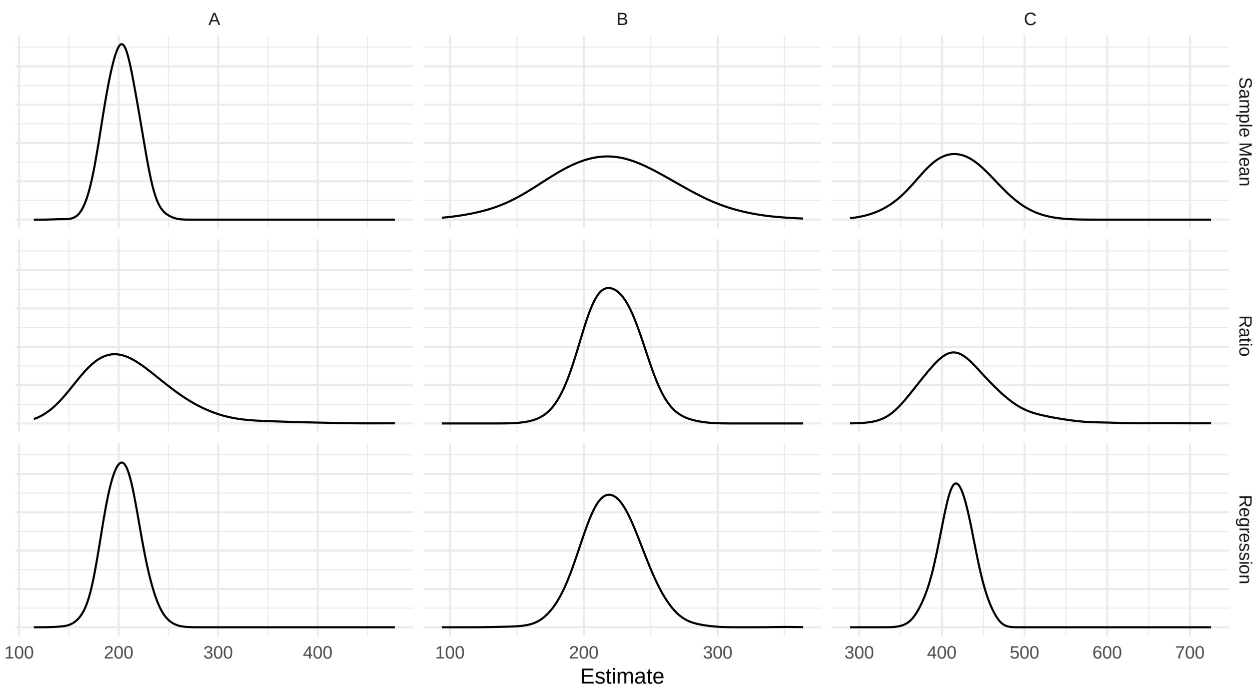
3. Assuming a simple random sampling design, what is the estimate and bound on the error of estimation using $\bar{y}$?

A Comparison of Three Estimators

In general, how do we expect the regression estimator to perform relative to the other two estimators?

Example: Consider the following simulation study with three populations and three estimators — the sample mean $\bar{y}$, the ratio estimator $\mu_x\bar{y}/\bar{x}$, and the regression estimator $\bar{y} + b(\mu_x - \bar{x})$ — applied to a sample of size $n = 10$ using simple random sampling.





Which estimator would we prefer when?

population	estimator	bias	variance	mse
A	Sample Mean	0.86	241.45	241.96
A	Ratio	8.63	2000.34	2072.77
A	Regression	1.19	279.27	280.40
B	Sample Mean	-1.24	1944.67	1944.26
B	Ratio	-0.63	391.13	391.13
B	Regression	-0.45	451.70	451.45
C	Sample Mean	-1.24	1737.24	1737.05
C	Ratio	7.23	2122.31	2172.50
C	Regression	-0.72	385.33	385.46

The Generalized Regression (GREG) Estimator

The generalized regression estimator of μ_y is

$$\hat{\mu}_y = \bar{y} + b_1(\mu_{x_1} - \bar{x}_1) + b_2(\mu_{x_2} - \bar{x}_2) + \cdots + b_k(\mu_{x_k} - \bar{x}_k)$$

where $\mu_{x_1}, \mu_{x_2}, \ldots, \mu_{x_k}$ are the population means of the k auxiliary variables.

The generalized regression estimator of $\hat{\tau}_y$ can be written as

$$\hat{\tau}_y = N\bar{y} + b_1(\tau_{x_1} - N\bar{x}_1) + b_2(\tau_{x_2} - N\bar{x}_2) + \cdots + b_k(\tau_{x_k} - N\bar{x}_k),$$

where $\tau_{x_1}, \tau_{x_2}, \ldots, \tau_{x_k}$ are the population totals of the k auxiliary variables. This can also be written as

$$\hat{\tau}_y = N\bar{y} + b_1(\tau_{x_1} - \hat{\tau}_{x_1}) + b_2(\tau_{x_2} - \hat{\tau}_{x_2}) + \cdots + b_k(\tau_{x_k} - \hat{\tau}_{x_k}),$$

where $\hat{\tau}_{x_1} = N\bar{x}_1, \hat{\tau}_{x_2} = N\bar{x}_2, \ldots, \hat{\tau}_{x_k} = N\bar{x}_k$.

The Prediction Perspective

We can write τ_y as

$$\tau_y = \sum_{i \in \mathcal{S}} y_i + \sum_{i \in \mathcal{S}'} y_i,$$

where $\mathcal{S}$ denotes the set of elements *included in the sample* and $\mathcal{S}'$ denotes the set of elements *excluded from the sample*.

Some estimators of τ_y take the form

$$\hat{\tau}_y = \sum_{i \in \mathcal{S}} y_i + \sum_{i \in \mathcal{S}'} \hat{y}_i,$$

where $\hat{y}_i$ denotes a *predicted value* of the target variable for an element that is not in the sample. Different estimators can be derived depending on how these predicted values are computed.

1. Let $\hat{y}_i = \bar{y}$, where $\bar{y}$ is the mean of the target variable for the elements in the sample. Then it can be shown that

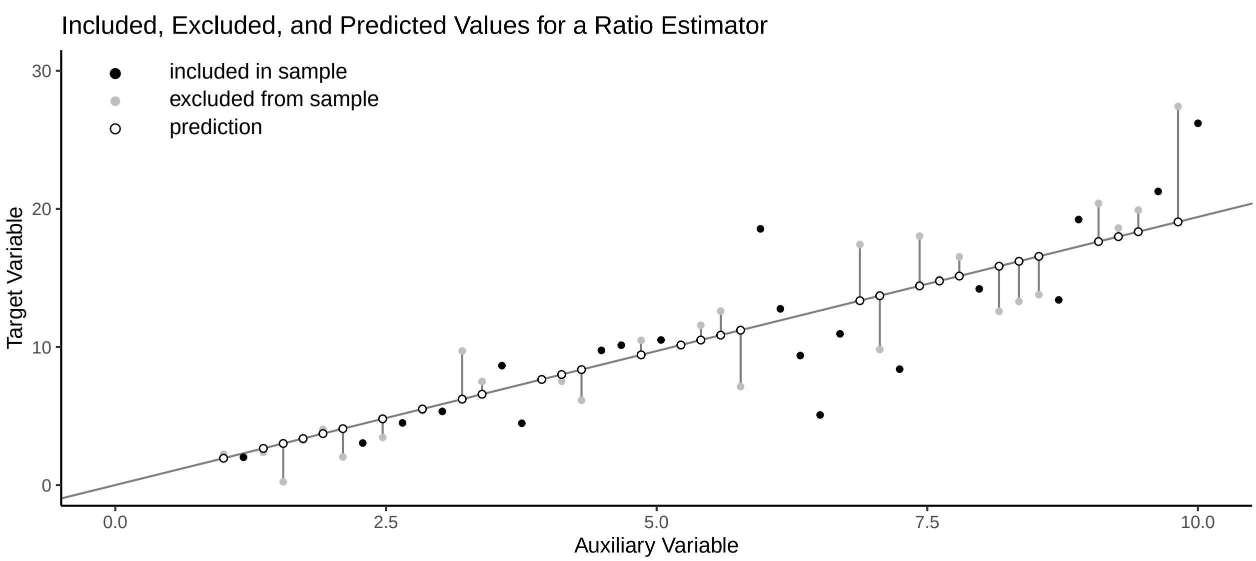
$$\hat{\tau}_y = \sum_{i \in \mathcal{S}} y_i + \sum_{i \in \mathcal{S}'} \bar{y} = N\bar{y}.$$

2. Let $\hat{y}_i = rx_i$ where $r = \bar{y}/\bar{x}$. Then it can be shown that

$$\hat{\tau}_y = \sum_{i \in \mathcal{S}} y_i + \sum_{i \in \mathcal{S}'} x_i\bar{y}/\bar{x} = \tau_x\bar{y}/\bar{x}.$$

3. Let $\hat{y}_i = a + bx_i$. Then it can be shown that

$$\hat{\tau}_y = \sum_{i \in \mathcal{S}} y_i + \sum_{i \in \mathcal{S}'} (a + bx_i) = N\bar{y} + b(\tau_x - N\bar{x}).$$



Note that the corresponding estimator of μ_y is then obtained as $\hat{\tau}_y/N$. That is,

$$\hat{\mu}_y = \frac{1}{N}\left(\sum_{i \in \mathcal{S}} y_i + \sum_{i \in \mathcal{S}'} \hat{y}_i\right).$$