Monday, Sep 23

Estimation of a Ratio and Ratio Estimators

Confusingly, there are two somewhat related problems that we will consider involving "ratios" in estimation.

1. The estimation of a *ratio of totals* of two variables (an "estimator of a ratio").
2. Using an estimated ratio to estimate a population *mean* or *total* (a "ratio estimator").

Today's lecture considers only the first problem — i.e., estimating a ratio of totals.

Estimation of a Ratio (of Totals)

Sometimes we are interested in estimating the *ratio* of the totals of *two* variables, defined as

$$R = \frac{\tau_y}{\tau_x} = \frac{\sum_{i=1}^N y_i}{\sum_{i=1}^N x_i}.$$

Note that because $\mu_y = \frac{1}{N}\sum_{i=1}^N y_i$ and $\mu_x = \frac{1}{N}\sum_{i=1}^N x_i$, then we can also write $R = \mu_y/\mu_x$.

Examples include the estimation of a *proportions*, *rates*, and *densities*.

Example: Consider the problem of estimating the *proportion* of trees that are of a particular species in a region of land divided into N plots.

Quantity	Description
y_i	number of trees of a particular species in plot i
x_i	number of trees in plot i
R	proportion of trees in the region that are of a particular species

Example: Consider the problem of estimating the birth *rate* in a population of N villages.

Quantity	Description
y_i	number of births in the past year in village i
x_i	number of people in village i
R	birth rate (number of births per person per year)

Example: Consider the problem of estimating the *density* of errors in a stream of data divided into N chunks of varying size.

Quantity	Description
y_i	number of errors in the chunk i
x_i	size of chunk i
R	error density (errors per unit size)

We will also see that sometimes estimators of means will take the form of a ratio. Examples include the estimation of the means of *domains*, and the estimation of means when elements are *clustered*.

Estimator for a Ratio

For a *simple random sampling* design, an estimator of R is

$$r = \frac{\hat{\tau}_y}{\hat{\tau}_x} = \frac{N\bar{y}}{N\bar{x}} = \frac{\bar{y}}{\bar{x}}.$$

Example: Consider a survey using a simple random sampling design with $n = 10$ villages to estimate birth rate, where y_i is the number of births in the i-th sampled village, and x_i is the number of people in the i-th sampled village.

village	y_i	x_i
a	6	58
b	7	158
c	5	82
d	9	177
e	6	188
f	1	9
g	4	106
h	4	178
i	4	110
j	8	91

We have that $\bar{y} = 5.4$ and $\bar{x} = 115.7$, so the estimated birth rate is then $r = 5.4/115.7 \approx 0.05$ births per person.

The estimated variance of the estimator r under simple random sampling is

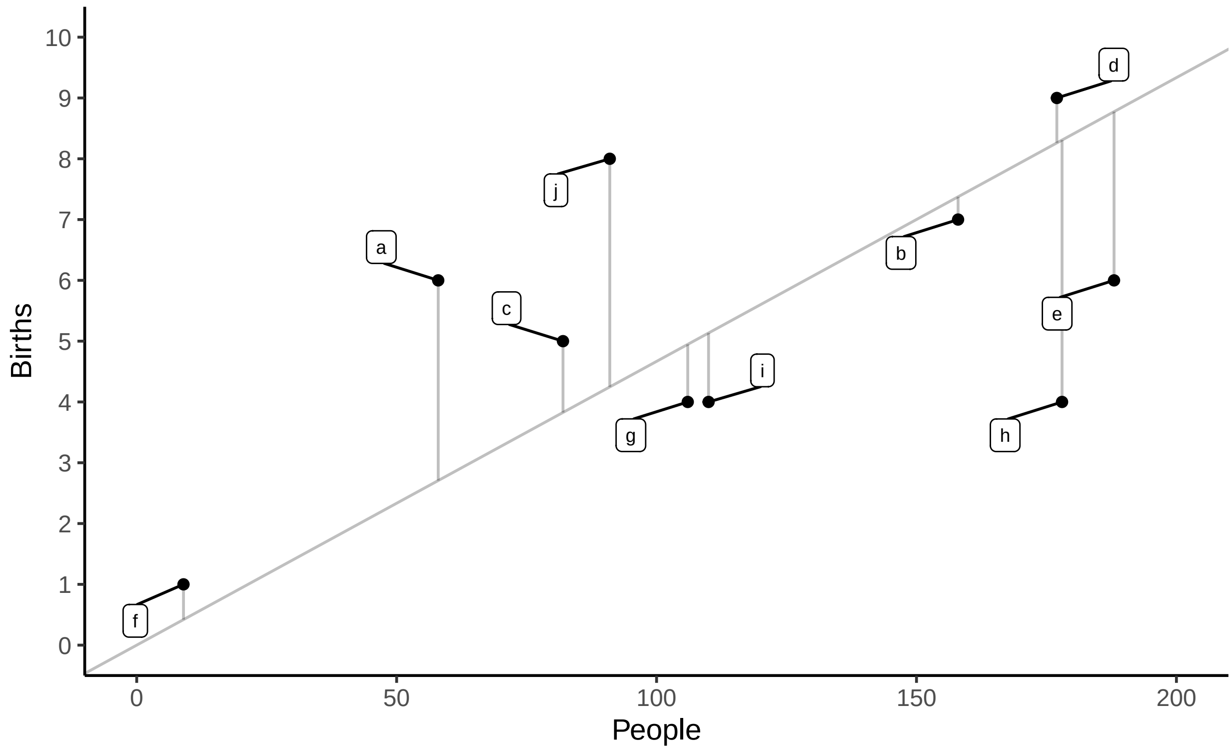
$$\hat{V}(r) = \left(\frac{1}{\mu_x^2}\right)\left(1 - \frac{n}{N}\right)\frac{s_r^2}{n} \quad \text{where} \quad s_r^2 = \frac{\sum_{i \in \mathcal{S}}(y_i - rx_i)^2}{n-1}.$$

Note: If μ_x is unknown it can be replaced with $\bar{x}$.

Example: Returning to the birth rate problem, what is $\hat{V}(r)$ if $N = 100$? And what is the bound on the error of estimation?

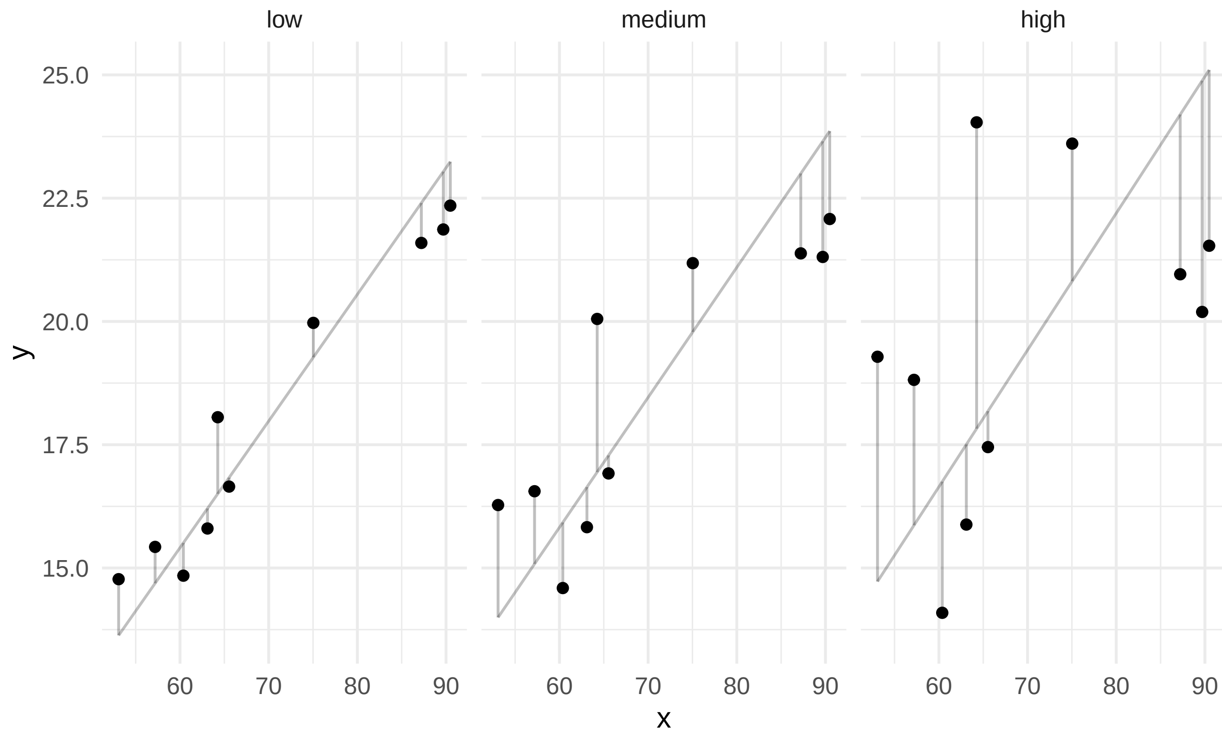
village	y_i	x_i	$y_i - rx_i$	$(y_i - rx_i)^2$
a	6	58	3.29	10.84
b	7	158	-0.37	0.14
c	5	82	1.17	1.38
d	9	177	0.74	0.55
e	6	188	-2.77	7.7
f	1	9	0.58	0.34
g	4	106	-0.95	0.9
h	4	178	-4.31	18.56
i	4	110	-1.13	1.29
j	8	91	3.75	14.08
				55.76

Note that quantities shown in the table are rounded.



We can show that $\hat{V}(r) \approx 0.00004165567$, which gives a bound on the error of estimation of $B \approx 0.013$.

How does the relationship between y_i and x_i affect the variance of r?

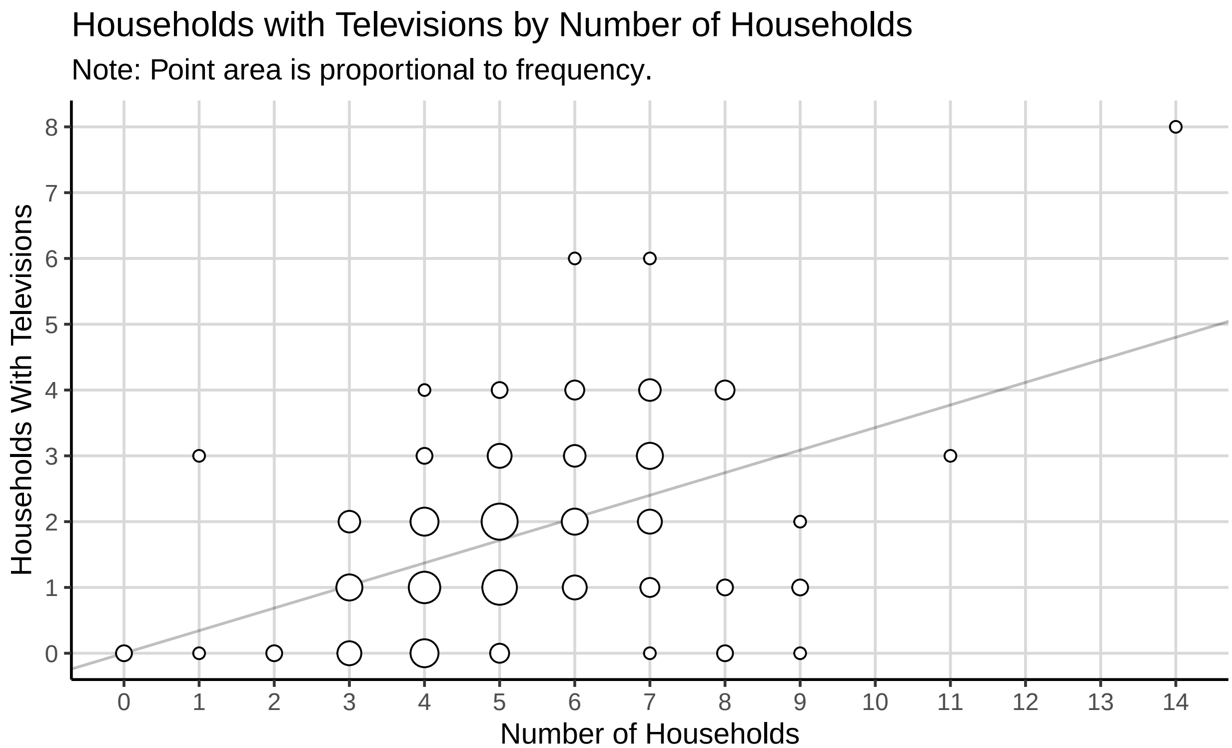


Note that the lines in the figures below have intercepts of zero (i.e., they pass through the origin) and slopes of r.

Example: Consider a survey to estimate the proportion of households with televisions in Des Moines, Iowa, in 1951. The sampling units are *blocks* of households. Here y_i is the number of households with televisions in the i-th block, and x_i is the number of households in the i-th block.

block	tv	households
1	1	3
2	0	8
3	1	8
4	0	7
5	4	7
⋮	⋮	⋮
132	2	4

We can compute $\bar{y} \approx 1.8$ and $\bar{x} \approx 5.23$. The estimate of the proportion of households with TVs is $r \approx 0.34$.



We have $\sum_{i \in \mathcal{S}}(y_i - rx_i)^2 \approx 210.63$, and we know that $N = 9460$ and that $\tau_x = 56296$ households, so $\mu_x = 56296/9460 \approx 5.95$. We can show that $\hat{V}(r) \approx 0.0003$ and that the bound on the error of estimation is $B \approx 0.04$.

Estimation of a μ With Clusters of Elements

Example: Consider a sampling design where the elements are plot, y_i is tree biomass in the i-th plot, and x_i is the number of trees in the i-th plot. Here are the data for a simple random sampling design that selected n = 10 plots.

Plot	Biomass	Trees
1	125	4
2	185	7
3	82	4
4	152	8
5	263	9
6	52	2
7	131	5
8	201	8
9	114	5
10	148	5

Note that mean volume *per tree* (μ) is

$$\mu = \frac{\sum_{i=1}^N y_i}{\sum_{i=1}^N x_i} = \frac{\text{total biomass for all trees}}{\text{total number of trees}},$$

which can be estimated by the ratio $r = \bar{y}/\bar{x}$. Here we have that $\hat{\mu} = r = \bar{y}/\bar{x} \approx 25.5$ units of biomass *per tree*.

Note: We will see this estimator again when we discuss *one-stage cluster sampling*.

Estimation of Domain Means

Under simple random sampling, the estimator of μ_d can be written as

$$\bar{y}_d = \frac{\sum_{i \in \mathcal{S}} y_i'}{\sum_{i \in \mathcal{S}} x_i},$$

where

$$y_i' = \begin{cases} y_i, & \text{if the } i\text{-th element is in the domain,} \\ 0, & \text{if the } i\text{-th element is not in the domain,} \end{cases}$$

and

$$x_i = \begin{cases} 1, & \text{if the } i\text{-th element is in the domain,} \\ 0, & \text{if the } i\text{-th element is not in the domain.} \end{cases}$$

Note that $\sum_{i \in \mathcal{S}} x_i = n_d$ (i.e., the number of elements in the sample that are in the domain).

Example: Here is how y_i' and x_i would be defined for a sample of $n = 10$.

The variance of $\bar{y}_d$ under simple random sampling tends to be a bit larger than if we had used the domains in a stratified random sampling design because n_d is *random* rather than *fixed by design*.

Domain	y_i	y'_i	x_i
yes	9	9	1
yes	5	5	1
yes	6	6	1
no	5	0	0
yes	2	2	1
no	8	0	0
no	3	0	0
no	2	0	0
yes	4	4	1
yes	9	9	1