Friday, September 19

Post-Stratification

Post-stratification is when stratification is *not* used in the design but *is* used in estimation.

1. Obtain a sample of size using a *simple random sampling*, *not* stratified random sampling.
2. Estimate the parameter (e.g., μ or τ) using the *estimator for stratified random sampling*.

Why post-stratification rather than stratified random sampling?

1. Stratification was an afterthought.
2. Stratified sampling was not possible.

The estimators for μ and τ for post-stratification are the same as those for stratified random sampling. They are

$$\hat{\mu} = \frac{N_1}{N}\bar{y}_1 + \frac{N_2}{N}\bar{y}_2 + \cdots + \frac{N_L}{N}\bar{y}_L = \sum_{j=1}^{L} \frac{N_j}{N}\bar{y}_j,$$

and

$$\hat{\tau} = N_1\bar{y}_1 + N_2\bar{y}_2 + \cdots + N_L\bar{y}_L = \sum_{j=1}^{L} N_j\bar{y}_j.$$

But the variances for these estimators are *not* the same under post-stratification and stratified random sampling. The variance of $\hat{\mu}$ when using simple random sampling with post-stratification is

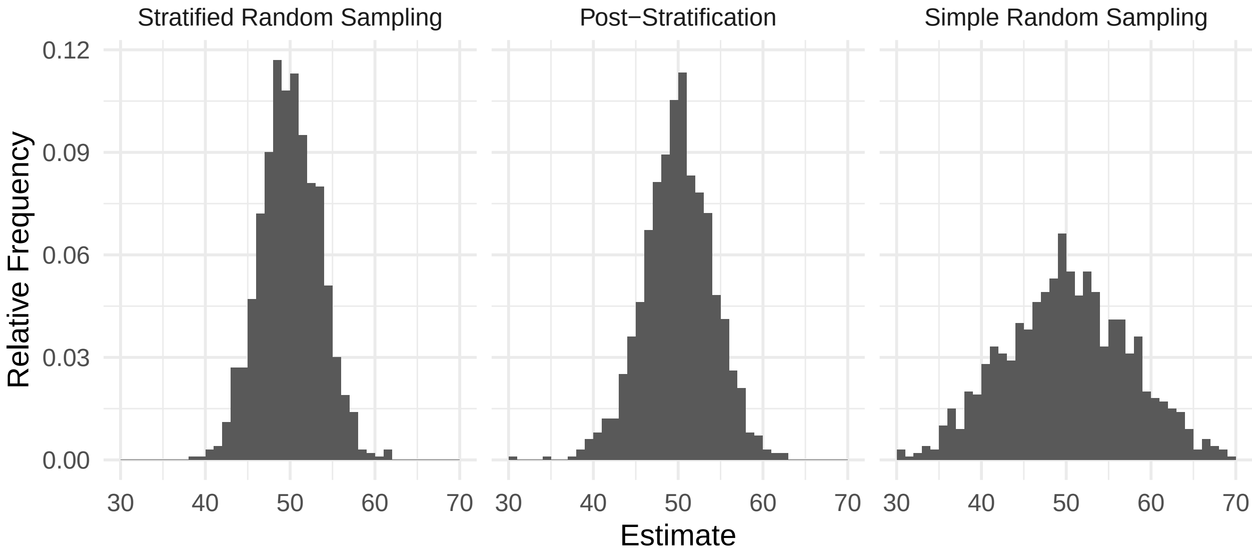
$$V(\hat{\mu}) \approx \underbrace{\frac{1}{N^2}\sum_{j=1}^{L} N_j^2\left(1 - \frac{n_j}{N}\right)\frac{\sigma_j^2}{n_j}}_{a} + \underbrace{\frac{1}{n^2}\left(\frac{N-n}{N-1}\right)\sum_{j=1}^{L}\left(1 - \frac{N_j}{N}\right)\sigma_j^2}_{b},$$

where $n_j = nN_i/N$ (proportional allocation). Recall that the variance of $\hat{\tau}$ is $V(\hat{\tau}) = N^2V(\hat{\mu})$.

1. The term a is the variance of $\hat{\mu}$ when using stratified random sampling with proportional allocation.
2. The term b is the extra variability due to random $n_1, n_2, \ldots, n_L$.

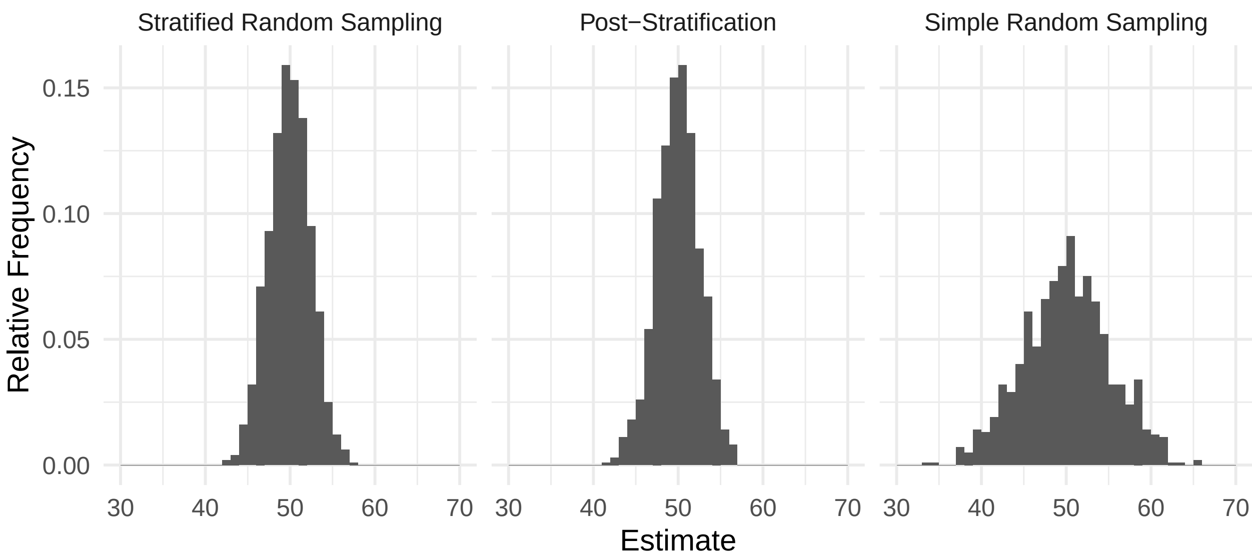
What does this imply about how stratified random sampling compares with post-stratification? And how does simple random sampling without post-stratification compare with simple random sampling with post-stratification?

Simulation results with three strata with $n = 15$.



Method	Variance
Stratified Random Sampling	3.48
Post-Stratification	4.40
Simple Random Sampling	7.33

Simulation results with three strata with $n = 30$.



Method	Variance
Stratified Random Sampling	2.44
Post-Stratification	2.56
Simple Random Sampling	5.13

Survey Weights

Recall that for a simple random sampling design, an estimator of τ is

$$\hat{\tau} = \frac{N}{n} \sum_{i \in \mathcal{S}} y_i,$$

which we can also write as $\hat{\tau} = N\bar{y}$. We can also write this as

$$\hat{\tau} = \sum_{i \in \mathcal{S}} w_i y_i,$$

where $w_i = N/n$. Furthermore it can be shown that $N = \sum_{i \in \mathcal{S}} w_i$.[1] So the estimator of μ (i.e., $\bar{y} = \hat{\mu}$) can be written as

$$\hat{\mu} = \frac{\sum_{i \in \mathcal{S}} w_i y_i}{\sum_{i \in \mathcal{S}} w_i}.$$

Every element has a **survey weight** w_i. More specifically, these are called **design weights** or **sampling weights** because they are determined by the sampling design. The weights can be interpreted as the effective number of elements in the population "represented" by the i-th sampled element.

Example: For a simple random sampling design with a population of $N = 100$ elements and a sample of $n = 20$ elements, what are the weights for each element in the sample?

Now consider a stratified random sampling design. Here the estimator of τ can be written as

$$\hat{\tau} = \hat{\tau}_1 + \hat{\tau}_2 + \cdots + \hat{\tau}_L,$$

where

$$\hat{\tau}_j = \frac{N_j}{n_j} \sum_{i \in \mathcal{S}_j} y_i,$$

where $\mathcal{S}_j$ is the sample obtained using simple random sampling form the j-th stratum, so we can write

$$\hat{\tau}_j = \sum_{i \in \mathcal{S}_j} w_i y_i,$$

where $w_i = N_j/n_j$. So the estimator of τ can also be written as

$$\hat{\tau} = \sum_{i \in \mathcal{S}} w_i y_i,$$

but now where $w_i = N_j/n_j$ if the i-th element in the j-th stratum.[2] As before, we can write the estimator $\hat{\mu}$ as

$$\hat{\mu} = \frac{\sum_{i \in \mathcal{S}} w_i y_i}{\sum_{i \in \mathcal{S}} w_i}.$$

Example: For a stratified random sampling design with two strata of sizes $N_1 = 200$ and $N_2 = 100$, and sample sizes of $n_1 = 10$ and $n_2 = 10$, what are the weights of the sampled elements?

[1] As there are n elements in the sample, the sum of the weights, $\sum_{i \in \mathcal{S}} \frac{N}{n}$, is equivalent to multiplying N/n by n which gives N.

[2] Another way to write this would be to use a sum within a sum as

$$\hat{\tau} = \sum_{j=1}^{L} \sum_{i \in \mathcal{S}_j} w_{ij} y_i,$$

where $w_{ij} = N_j/n_j$.

The weights depend on the design.

1. For simple random sampling, all observations have identical weights of N/n.
2. For stratified random sampling, all observations *within* a stratum j have identical weights of N_j/n_j. But weights are *not* necessarily the same *across* strata.

Re-Weighting

For various reasons we may decide to *change* the weights. This is called **re-weighting**.

Post-stratification can be viewed as re-weighting. If we use a simple random sampling design we *could* use the estimator

$$\hat{\tau} = \sum_{i \in \mathcal{S}} w_i y_i,$$

where $w_i = N/n$. But when using post-stratification we use our knowledge of the stratification of the elements to *re-weight* by replacing these weights with those from a stratified random sampling design where now $w_i = N_j/n_j$ if the i-th element is known to be within the j-th stratum.