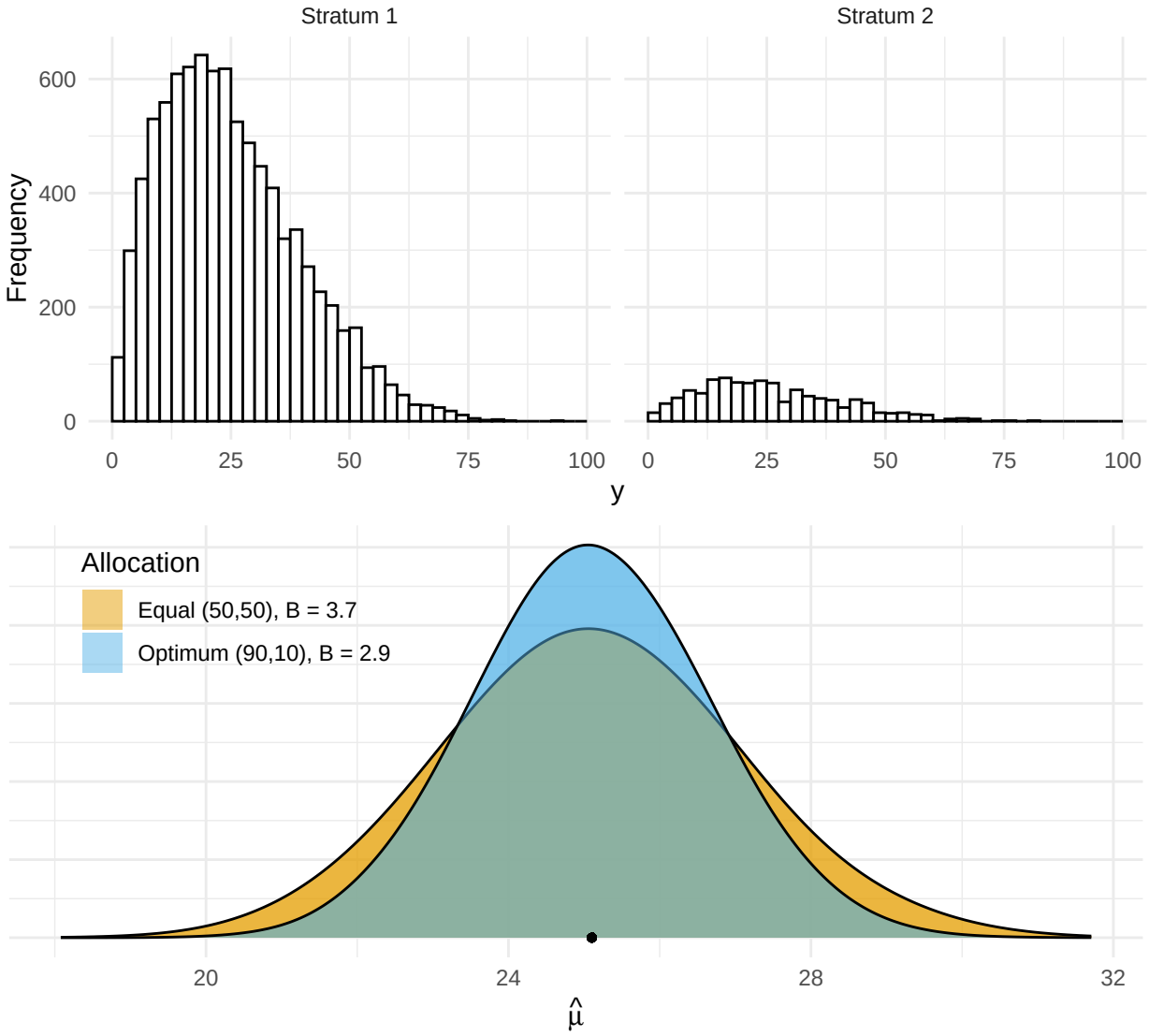
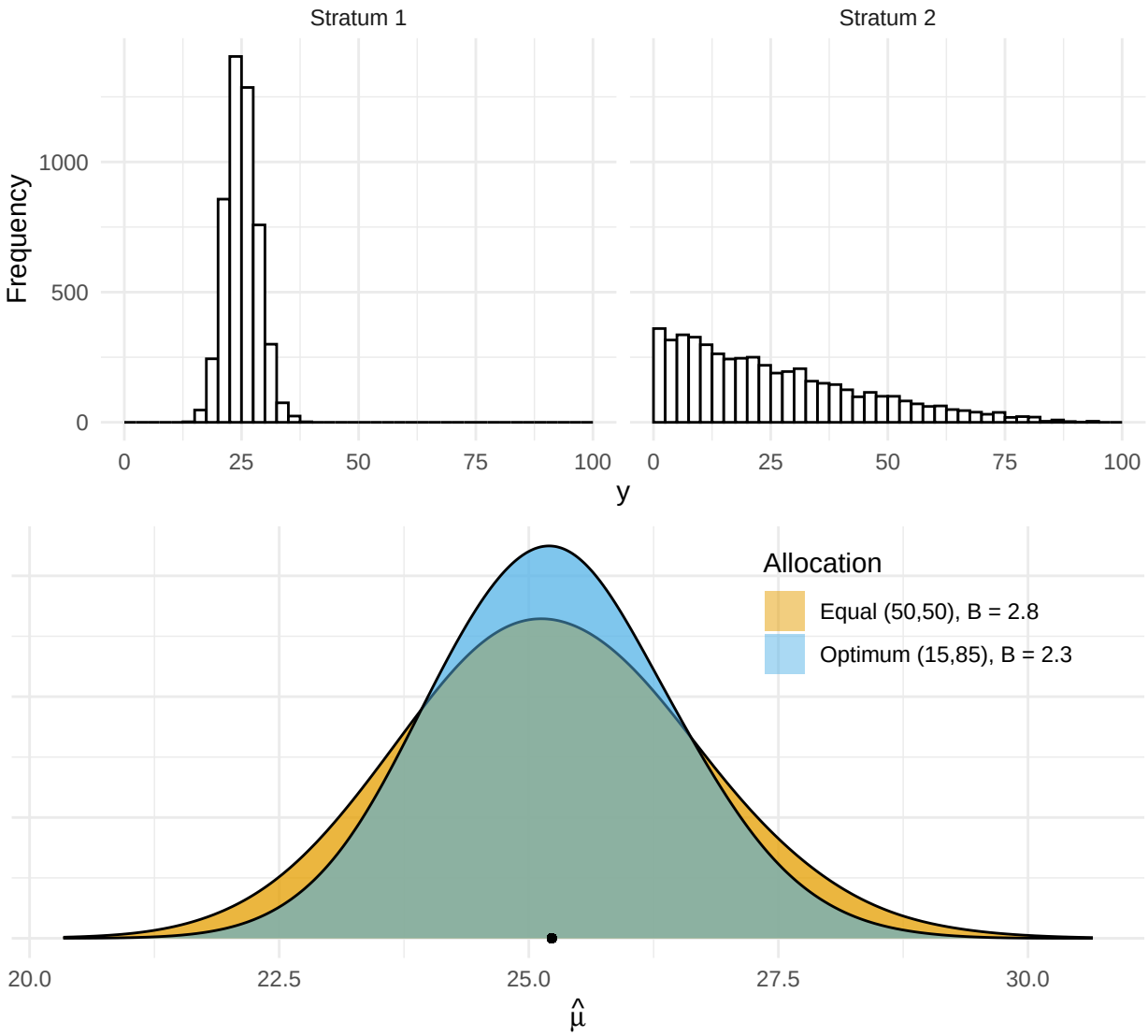
# Monday, Sep 9

### Simulation Study of Optimum Allocation

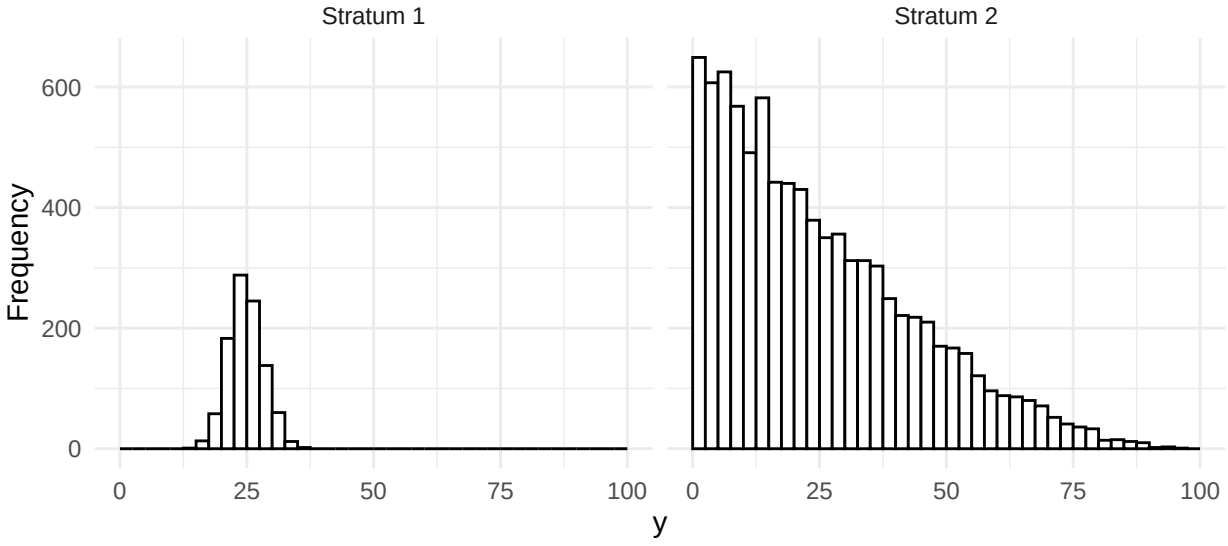
Strata have same variance, but different sizes.

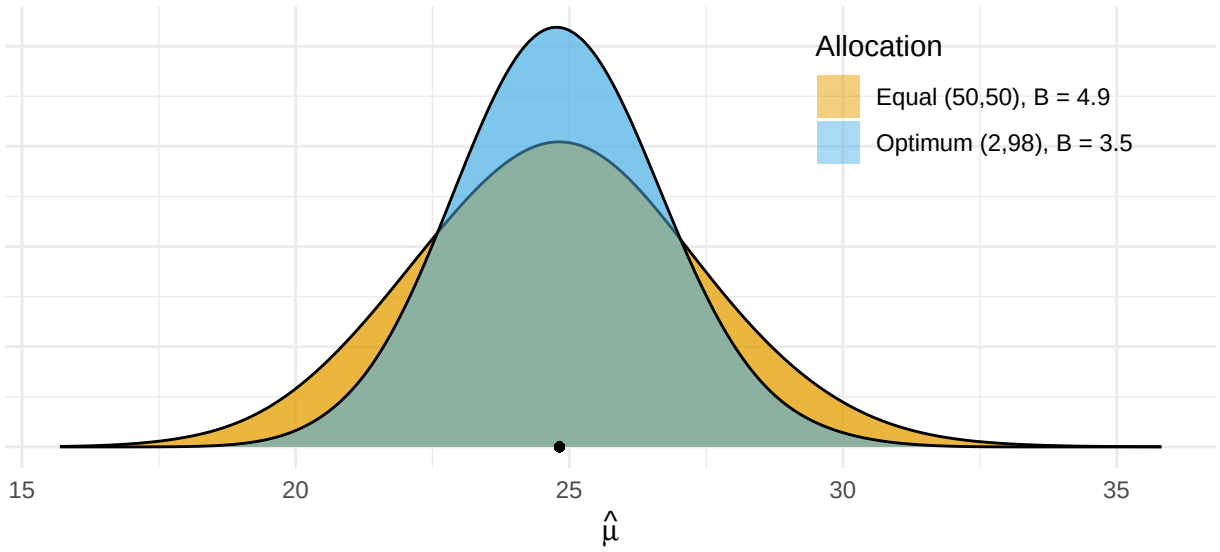



Strata have different variance, but same sizes.




Strata have different variances and sizes.






## Comparison of Stratified and Simple Random Sampling

How does stratified random sampling compare with simple random sampling with respect to the variance of the estimators?

### Analytical Results

Let $V_{\text{Opt}}$ and $V_{\text{Prop}}$ represent the variance of the estimator $\hat{\tau}$ or $\hat{\mu}$ from a stratified random sampling design with optimum and proportional allocation, respectively. And let $V_{\text{SRS}}$ represent the variance of the estimator for a simple random sampling design. It can be shown that generally

$$V_{\text{SRS}} \geq V_{\text{Prop}} \geq V_{\text{Opt}}.$$

But *how* different are these variances?

1. The difference between *proportional* and *optimum* allocation stratified random sampling designs is

$$V_{\text{Prop}} - V_{\text{Opt}} = \frac{1}{n}\sum_{j=1}^{L}\frac{N_j}{N}(\sigma_j - \bar{\sigma})^2, \quad \text{where} \quad \bar{\sigma} = \sum_{j=1}^{L}\frac{N_j}{N}\sigma_j.$$

   So when is optimum allocation substantially better than proportional allocation in stratified random sampling?

2. The difference between *simple random sampling* and *proportional allocation stratified random sampling* is

$$V_{\text{SRS}} - V_{\text{Prop}} \approx \frac{1}{n}\left(1 - \frac{n}{N}\right)\sum_{j=1}^{L}\frac{N_j}{N}(\mu_j - \mu)^2.$$

   So when is proportionally-allocated stratified random sampling substantially better than simple random sampling?

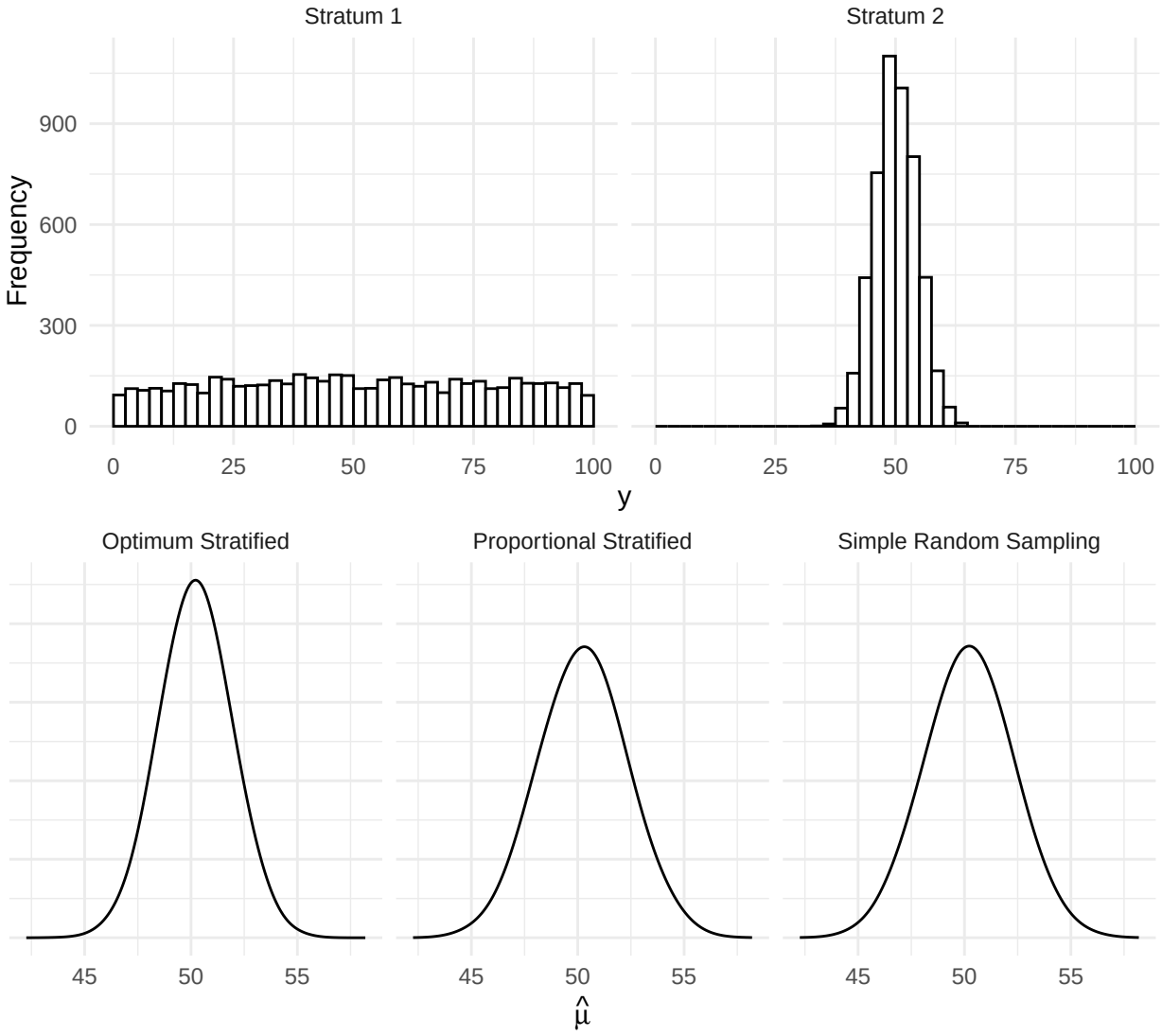
3. The difference between *simple random sampling* and *optimum-allocated stratified random sampling* is

$$V_{\text{SRS}} - V_{\text{Opt}} \approx \frac{1}{n}\sum_{j=1}^{L}\frac{N_j}{N}(\sigma_j - \bar{\sigma})^2 + \frac{1}{n}\left(1 - \frac{n}{N}\right)\sum_{j=1}^{L}\frac{N_j}{N}(\mu_j - \mu)^2.$$

So when is optimum-allocated stratified random sampling substantially better than simple random sampling?

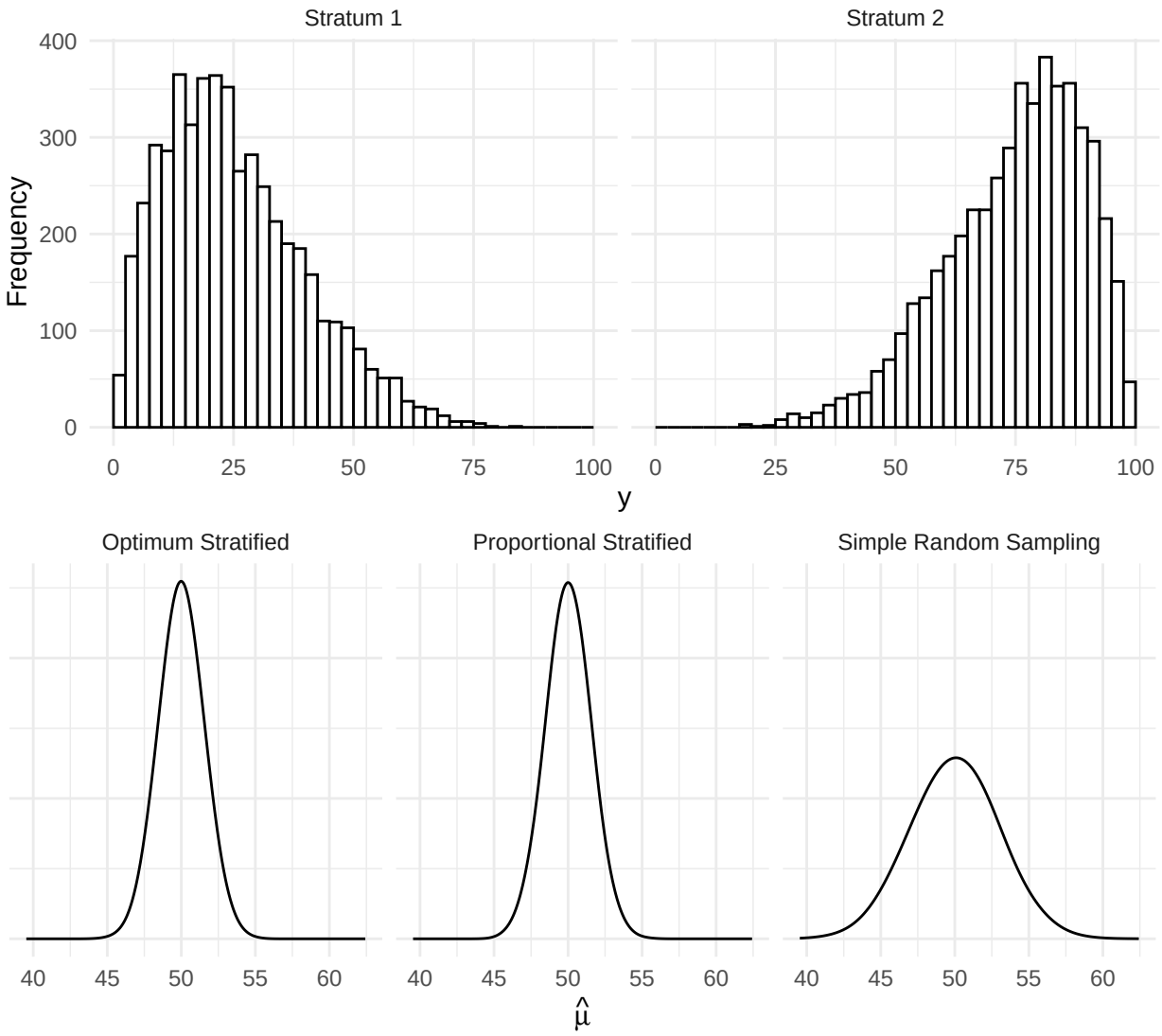
**Simulation Results**

Strata have *same* means, *different* variance.



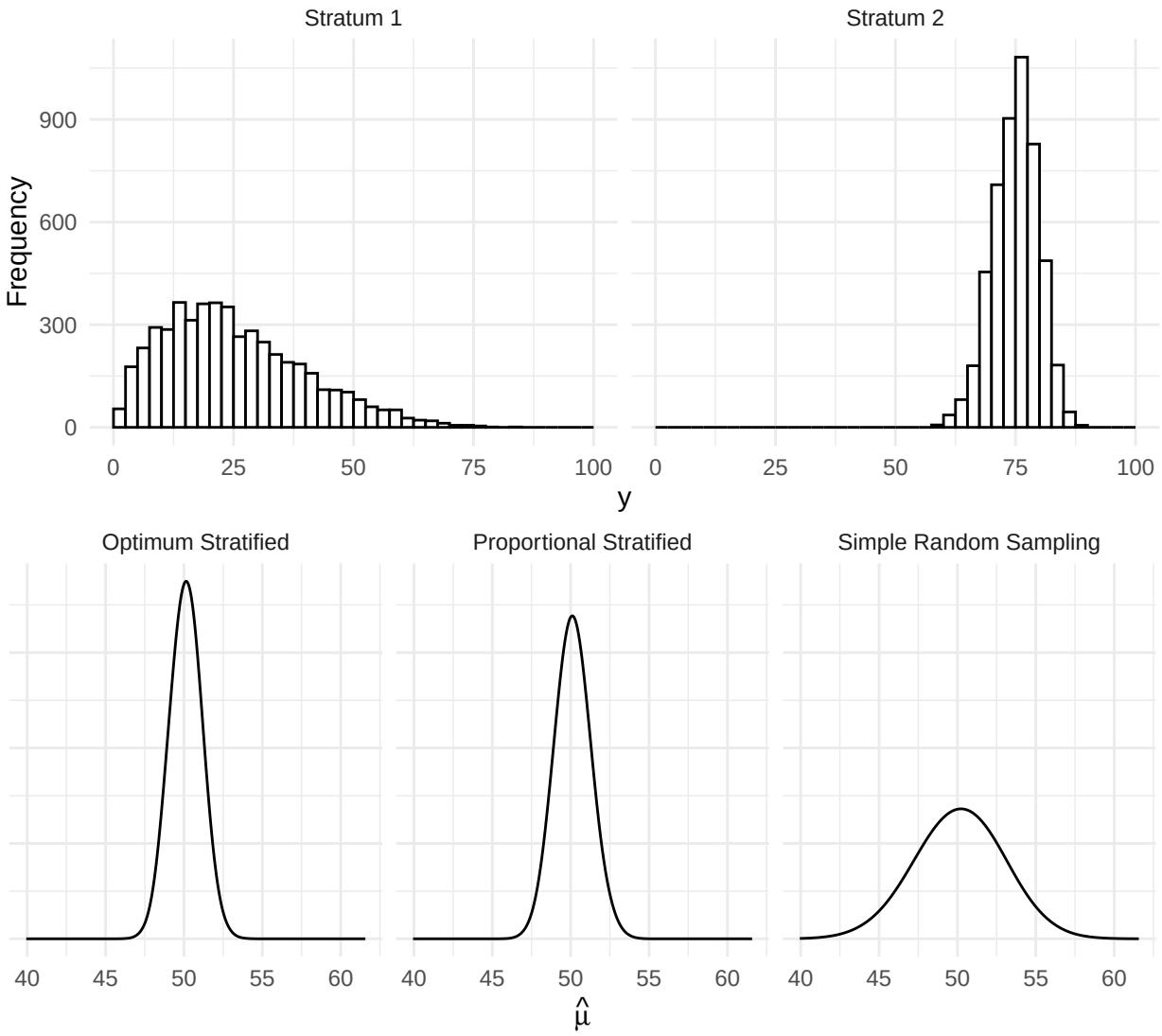

| design | V | SE | B |
|---|---|---|---|
| Optimum Stratified | 2.61 | 1.62 | 3.24 |
| Proportional Stratified | 3.90 | 1.97 | 3.94 |
| Simple Random Sampling | 3.91 | 1.98 | 3.96 |

Strata have *different* means, *same* variance.




| design | V | SE | B |
|---|---|---|---|
| Optimum Stratified | 2.09 | 1.45 | 2.90 |
| Proportional Stratified | 2.11 | 1.45 | 2.90 |
| Simple Random Sampling | 8.05 | 2.84 | 5.68 |

Strata have *different* means, *different* variance.




| design | V | SE | B |
|---|---|---|---|
| Optimum Stratified | 0.93 | 0.96 | 1.92 |
| Proportional Stratified | 1.18 | 1.09 | 2.18 |
| Simple Random Sampling | 7.18 | 2.68 | 5.36 |

## Design Effect

The **design effect** of a complex sampling design is defined as

$$D = \frac{V_{\mathrm{C}}}{V_{\mathrm{SRS}}}$$

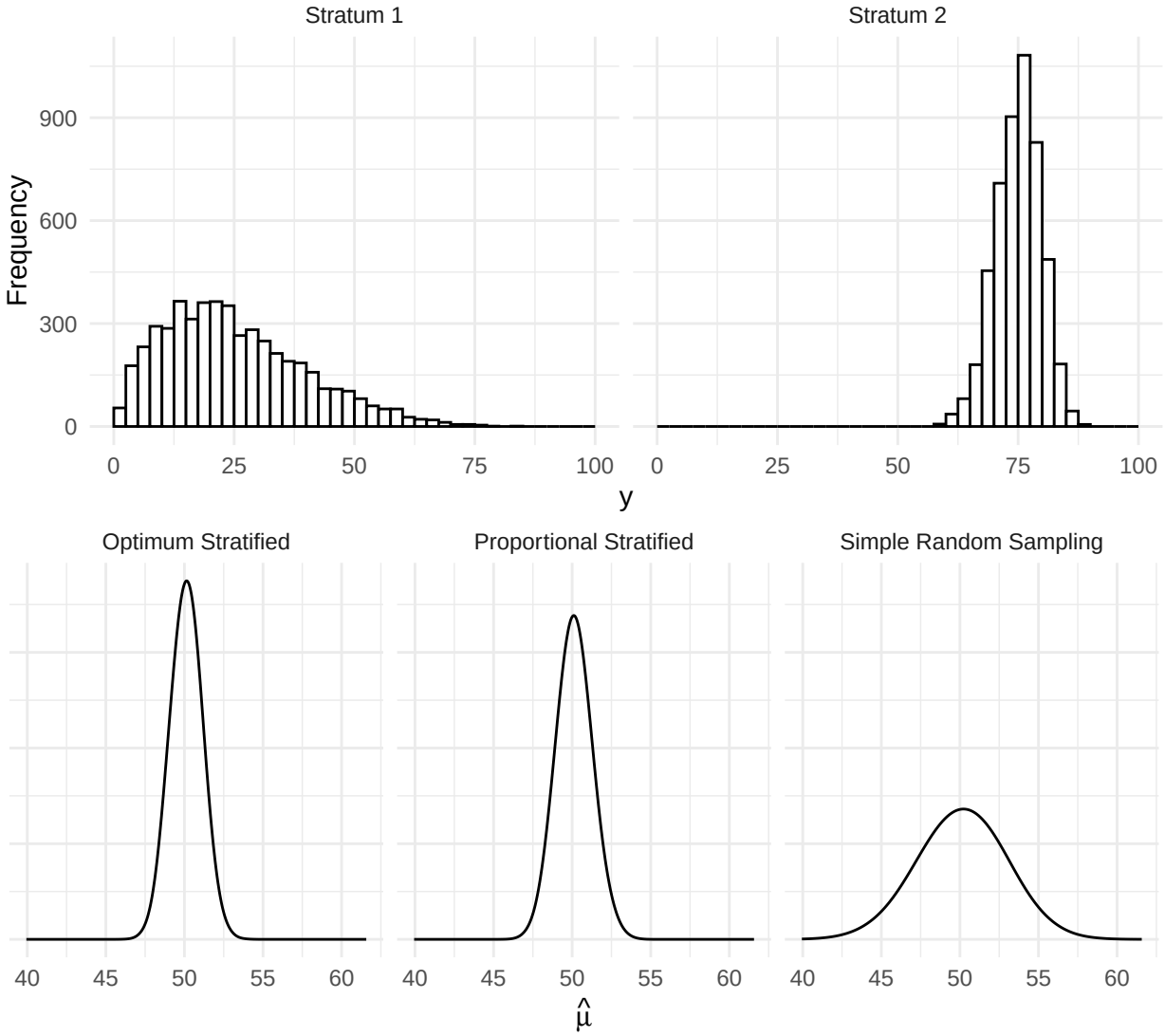
where $V_C$ and $V_{\mathrm{SRS}}$ are the variances of the estimators for a paramater under a complex sampling design (e.g., stratified random sampling) and simple random sampling, respectively. Clearly

$$\begin{aligned} &D < 1 \Leftrightarrow V_{\mathrm{C}} < V_{\mathrm{SRS}} \quad \text{(i.e., complex is better)},\\ &D = 1 \Leftrightarrow V_{\mathrm{C}} = V_{\mathrm{SRS}} \quad \text{(i.e., same)},\\ &D > 1 \Leftrightarrow V_{\mathrm{C}} > V_{\mathrm{SRS}} \quad \text{(i.e., SRS is better)}. \end{aligned}$$

The design effect can be computed a couple of ways.

1. Hypothetical analysis (using equations or simulation).
2. Estimated from a sample drawn using the complex sampling design.

**Example**: Recall the simulation with the strata with unequal means and variances.

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| Optimum Stratified | 0.93 | 0.96 | 1.92 |
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From this simulation we have that $V_{\text{Opt}} \approx 0.93$, $V_{\text{Prop}} \approx 1.18$, and $V_{\text{SRS}} \approx 7.18$. What are the design effects of the two stratified random sampling designs?

## Effective Sample Size

The **effective sample size** of a complex sampling design is defined as

$$\text{ESS} = \frac{n}{D},$$

and is interpreted as the sample size that a simple random sampling design would need for an estimator to have the same variance as that based on a complex sampling design.

**Example**: The sample size used in the simulation above is $n = 100$. What is the *effective sample size* of the two stratified random sampling designs?