Wednesday, Jan 18

Summary Measures of a Distribution — Continued

A couple of properties of a distribution that we often want to measure are *location* and *variability*.

Measures of Variability

The **variance** for a sample of observations can be written as

$$s^2 = \frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n-1} = \frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}.$$

Example: Consider the following sample of observations: 1, 1, 7. The mean is $\bar{x} = 3$ and the variance is

$$s^2 = \frac{(1-3)^2 + (1-3)^2 + (7-3)^2}{3-1} = 12.$$

A related measure is the **standard deviation** which is the square root of the variance, so it can be written as

$$s = \sqrt{\frac{(x_1 - \bar{x})^2 + (x_2 - \bar{x})^2 + \cdots + (x_n - \bar{x})^2}{n-1}} = \sqrt{\frac{\sum_{i=1}^n (x_i - \bar{x})^2}{n-1}}.$$

Note that the symbol for the variance is s^2 because the variance equals the square of the standard deviation (s).

Another measure is the **range** which is simply defined as the difference between the largest and smallest values,

$$\text{range} = \max(x_1, x_2, \ldots, x_n) - \min(x_1, x_2, \ldots, x_n),$$

and the **interquartile range** which we will discuss later.

Cumulative Distributions

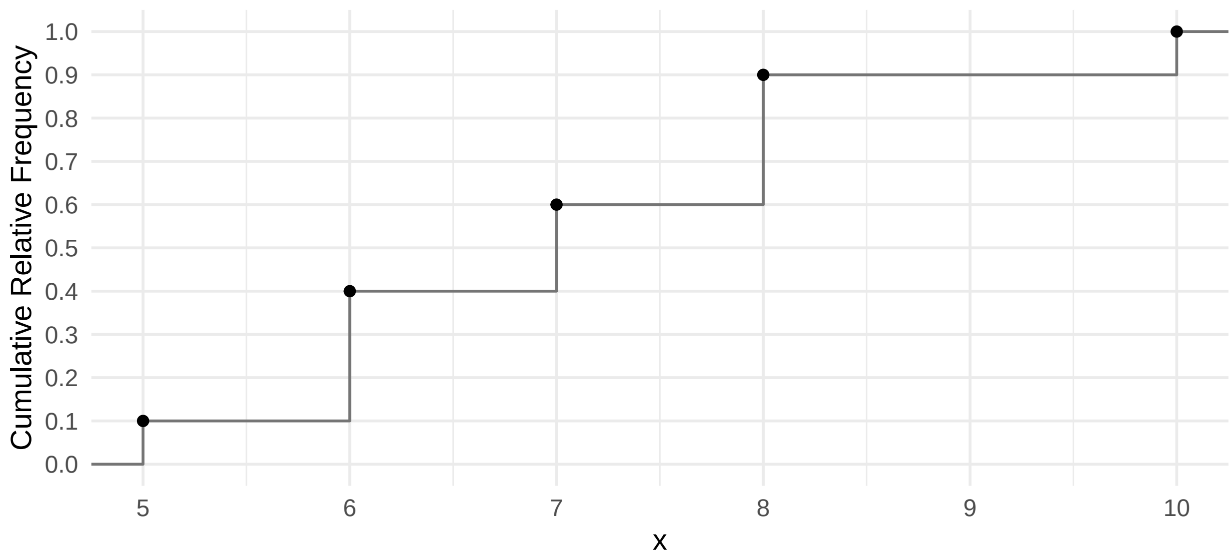
The **cumulative distribution** shows the relationship between the value of the variable and *cumulative relative frequency*.

Example: The following is a hypothetical set of observations of examination scores.

5, 6, 6, 6, 7, 7, 8, 8, 8, 10

x	Frequency	Relative Frequency	Cumulative Relative Frequency
5	1	0.1	0.1
6	3	0.3	0.4
7	2	0.2	0.6
8	3	0.3	0.9
10	1	0.1	1.0

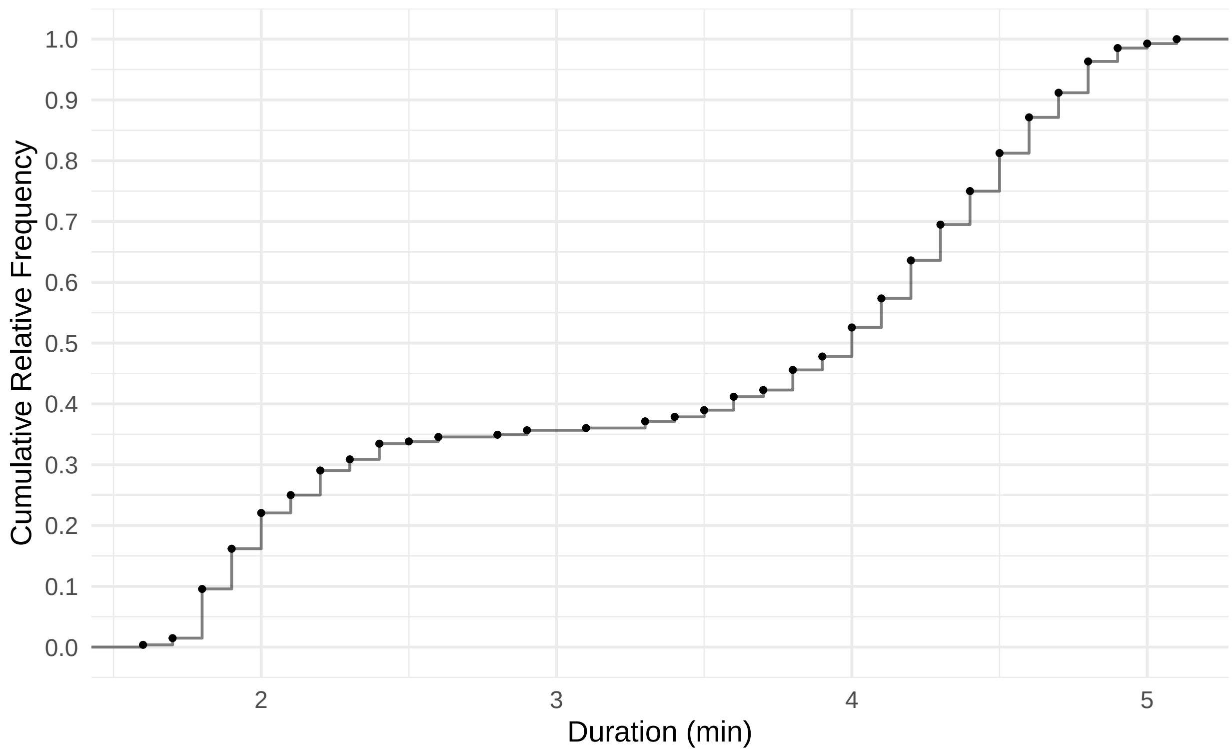
We can graph the cumulative distribution using a *step function*.



Example: Consider the cumulative distribution of the sample of observations of eruption duration of Old Faithful.

Time	Frequency	Relative Frequency	Cumulative Relative Frequency
1.6	1	0.004	0.004
1.7	3	0.011	0.015
1.8	22	0.081	0.096
1.9	18	0.066	0.162
2	16	0.059	0.221
⋮	⋮	⋮	⋮
5.1	2	0.007	1

Note: The relative and cumulative relative frequencies above have been rounded.



Finding Percentiles Using a Cumulative Distribution

The Pth **percentile** is the value of the variable such that $P\%$ of the observations are *less than that value*.

Finding Percentiles: Finding percentiles from a set of observations is surprisingly complex! Consider the following distribution.

Here are a couple of examples.

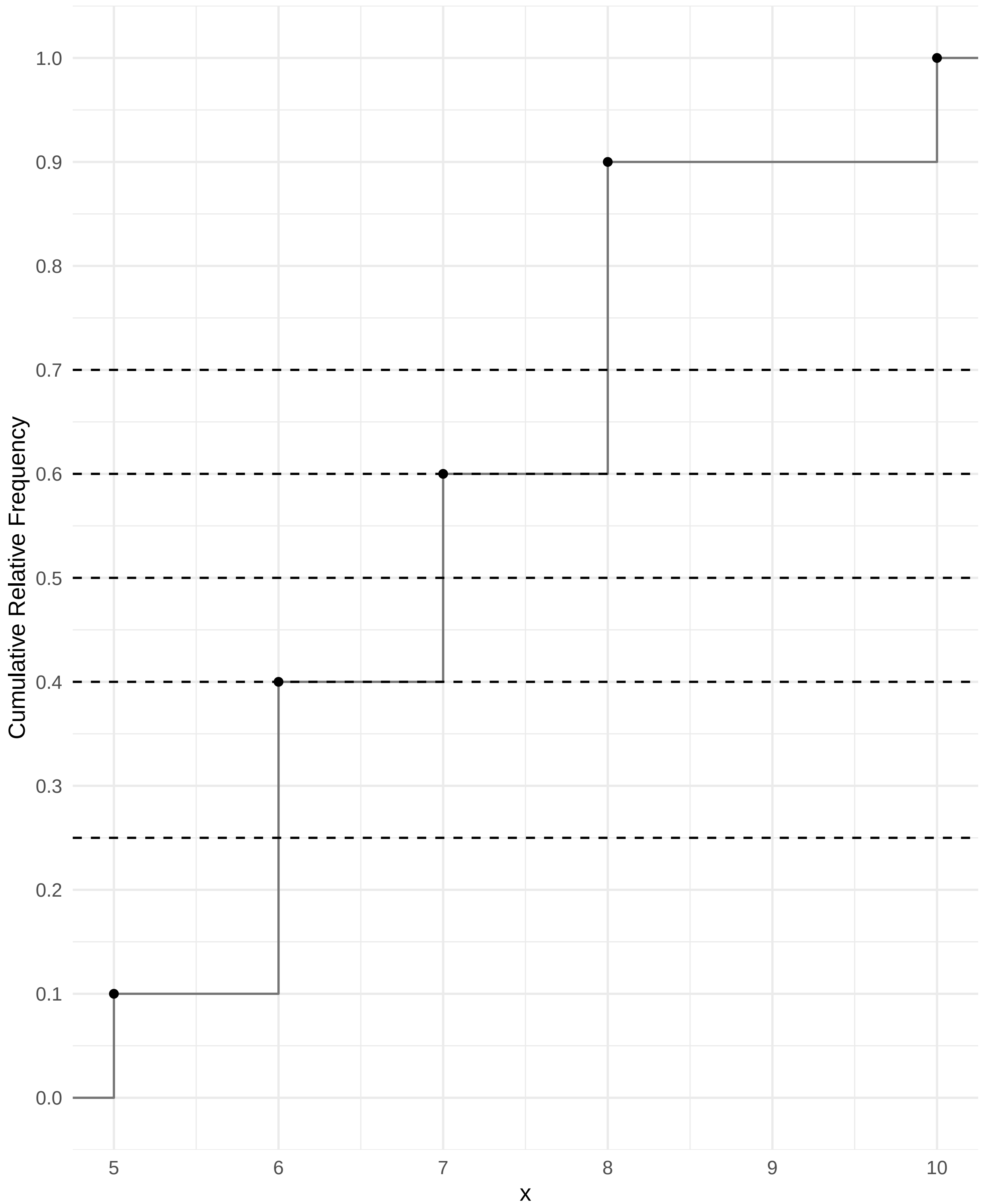
1. What is the 60th percentile? *Several* values of x would qualify! Any x such that $7 < \text{x} \leq 8$ has 60% of the observations that are less than it.
2. What is the 70th percentile? *No* values of x would qualify! The percent of observations less than 8 is 60%, and the percent of observations for any $x > 8$ is 90% or more.

x	Frequency	Relative Frequency	Cumulative Relative Frequency
5	1	0.1	0.1
6	3	0.3	0.4
7	2	0.2	0.6
8	3	0.3	0.9
10	1	0.1	1.0

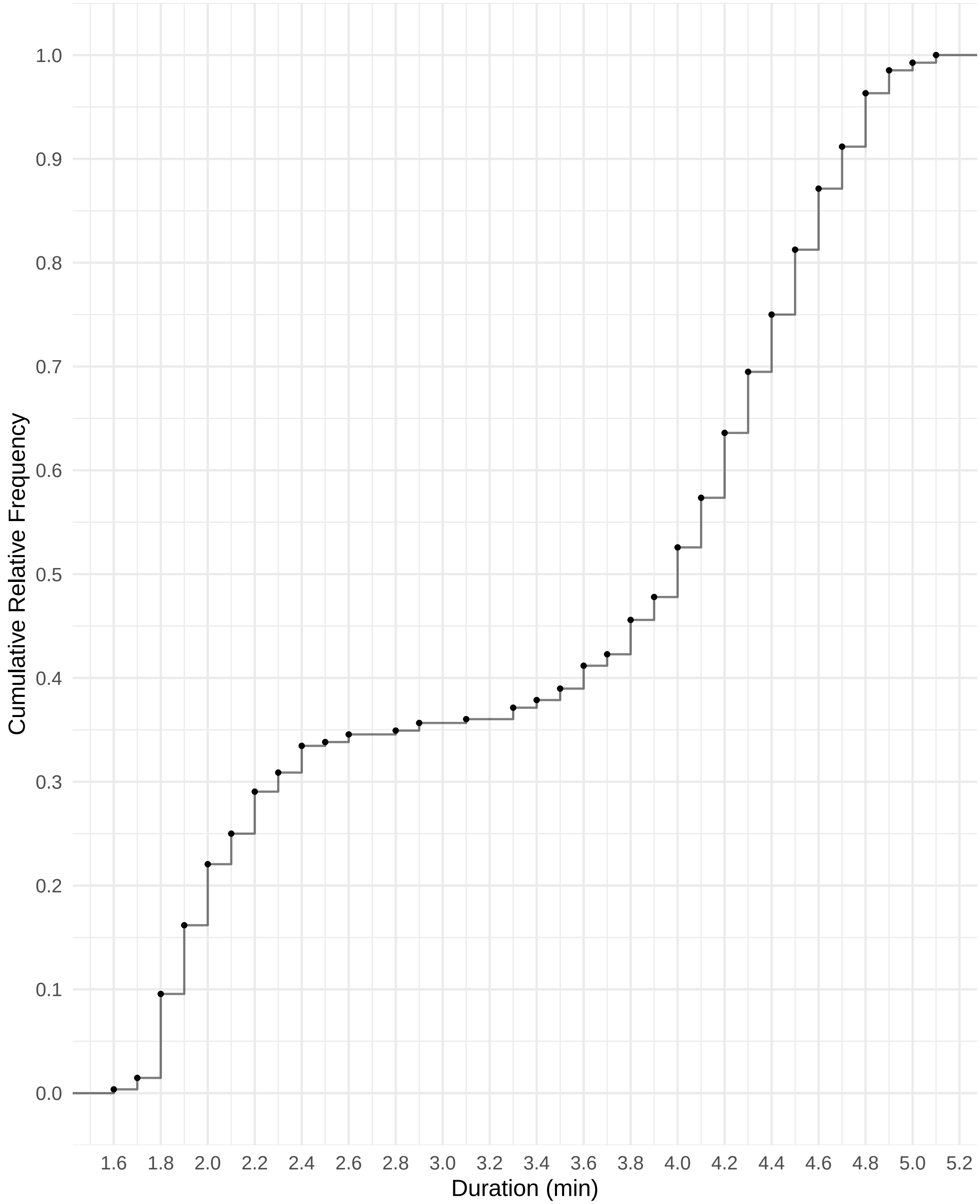
One solution is to use the *midpoint in the first case*, and *the smallest value of x that has more than P% of observations less than it* in the second case. This is easier to explain/do using a graph of the cumulative distribution.

Finding Percentiles Using the Cumulative Distribution: To find the approximate percentile using a *graph* of the cumulative relative distribution, find the value where the step function crosses the cumulative relative frequency of $P/100$. If more than one value crosses value, use the midpoint (i.e., average of the two values).

Example: We can confirm that the 25th, 40th, 50th, 60th, and 70th percentiles for the distribution show below are 6, 6.5, 7, 7.5, and 8, respectively.



Example: We can use the following plot to approximate the 25th, 50th, and 75th percentiles "by eye" (the actual percentiles are 2.15, 4, and 4.45).



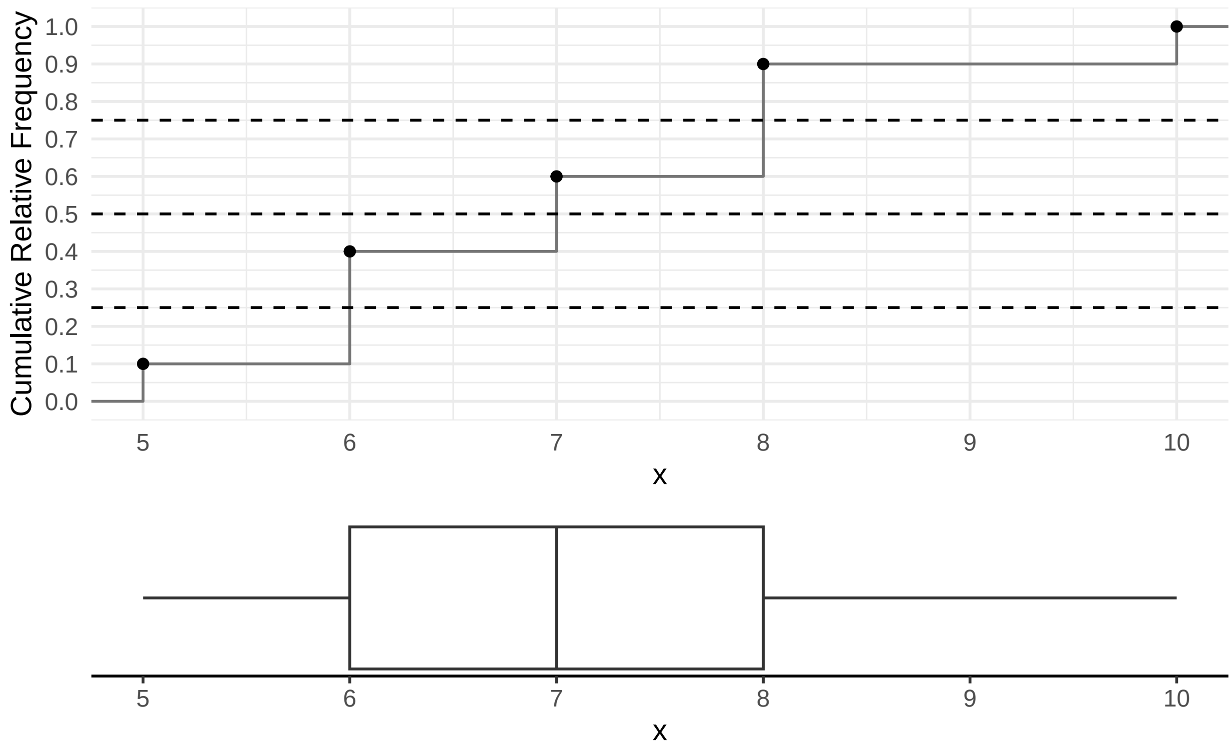
Box Plots

A **box plot** is a graphical representation of a distribution of a quantitative variable that uses what is called a **five-number summary**:

1. minimum
2. first quartile (Q_1) — i.e., 25th percentile
3. second quartile (Q_2) — i.e., 50th percentile and median
4. third quartile (Q_3) — i.e., 75th percentile
5. maximum

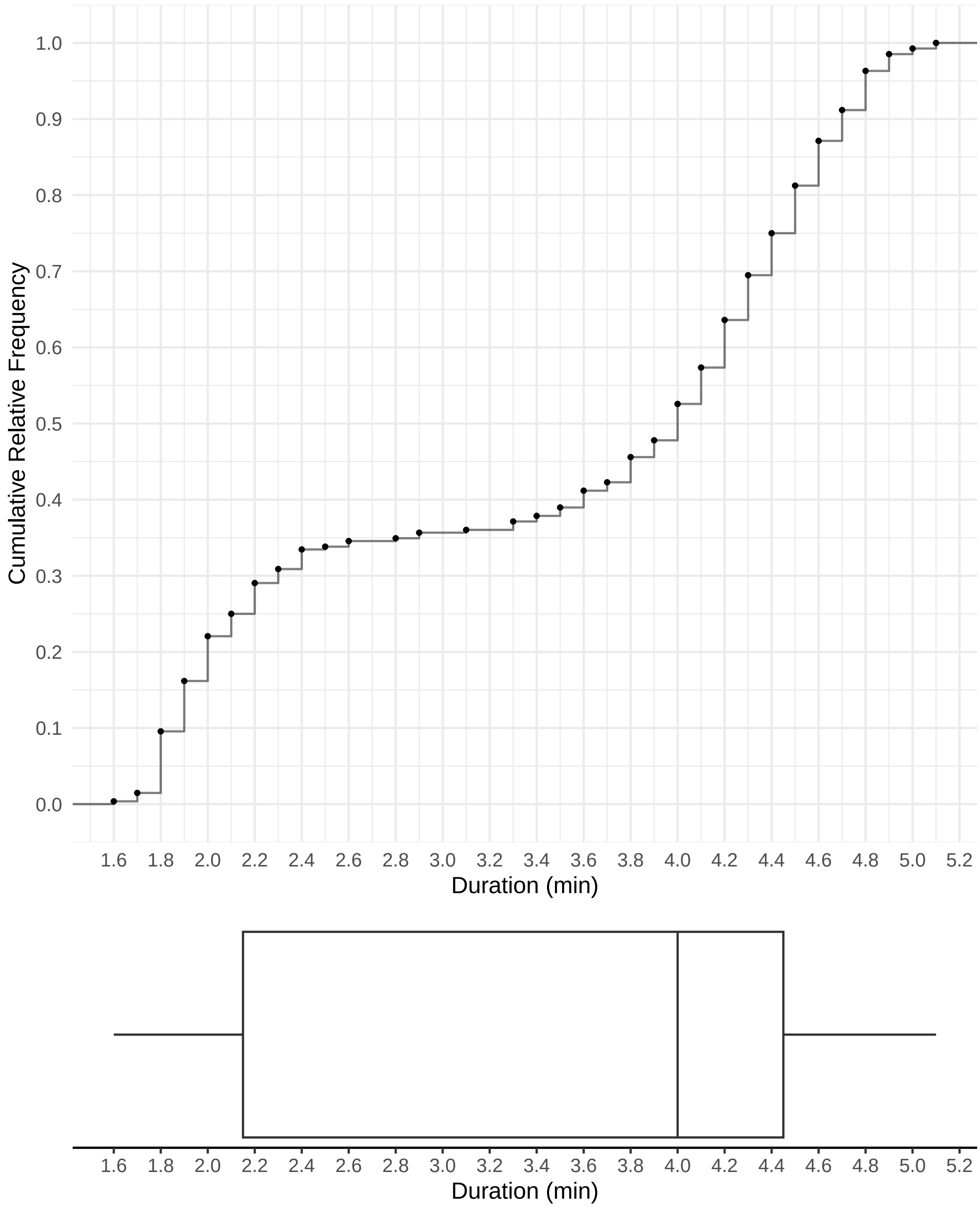
Comment: Because there is more than one way to approximate a percentile and thus a quartile, there is more than one way to draw a boxplot. For consistency we will use the approximation given earlier for finding percentiles.

Example: Box plot based on an earlier example.



Note that the five number summary is 5, 6, 7, 8, and 10.

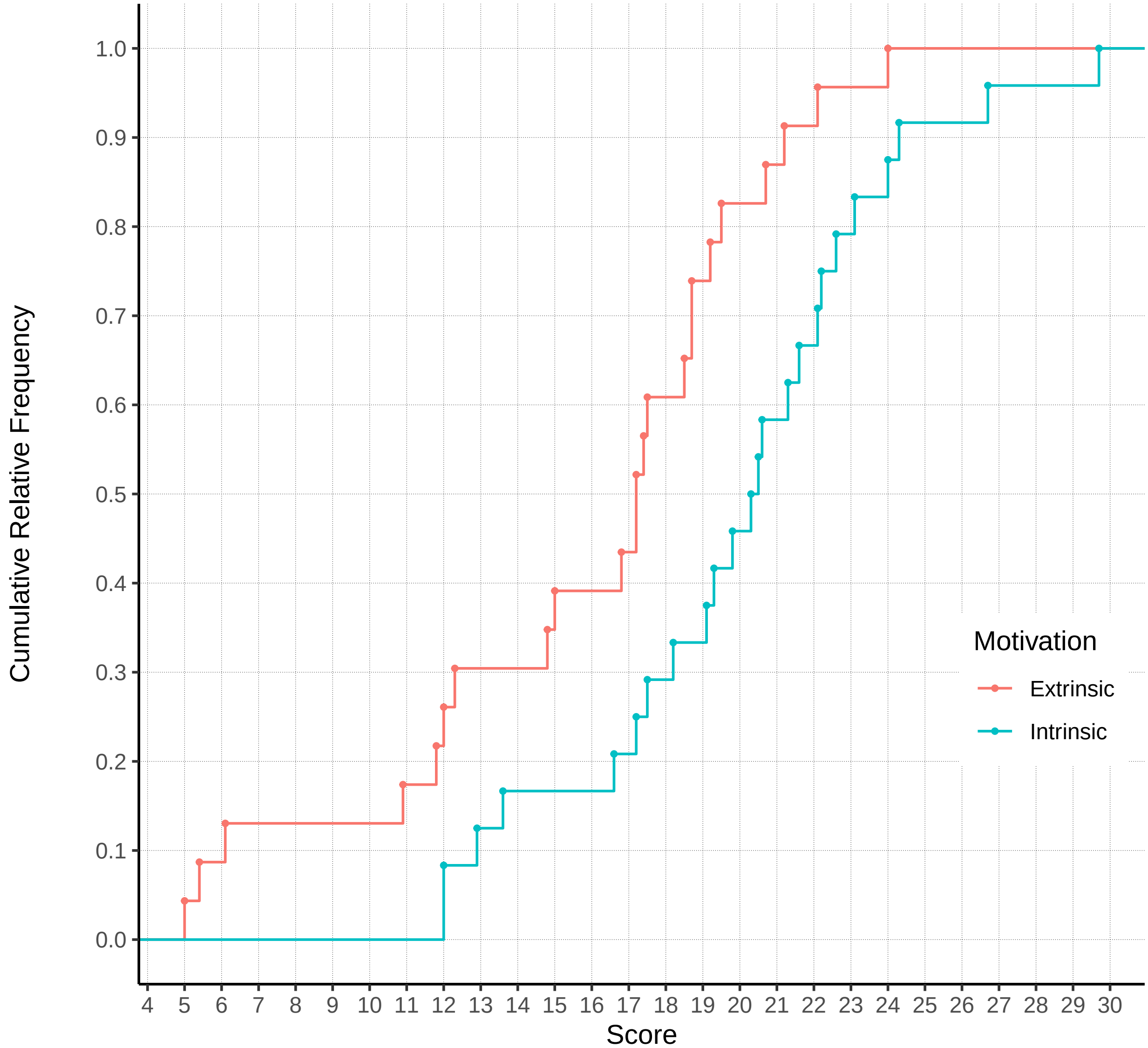
Example: Box plot of the Old Faithful data.

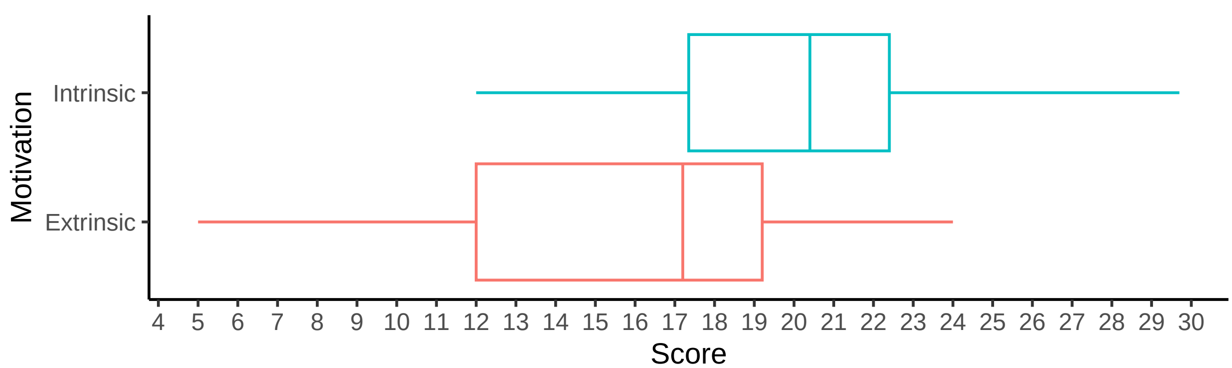


Note that the five number summary is 1.6, 2.15, 4, 4.45, and 5.1.

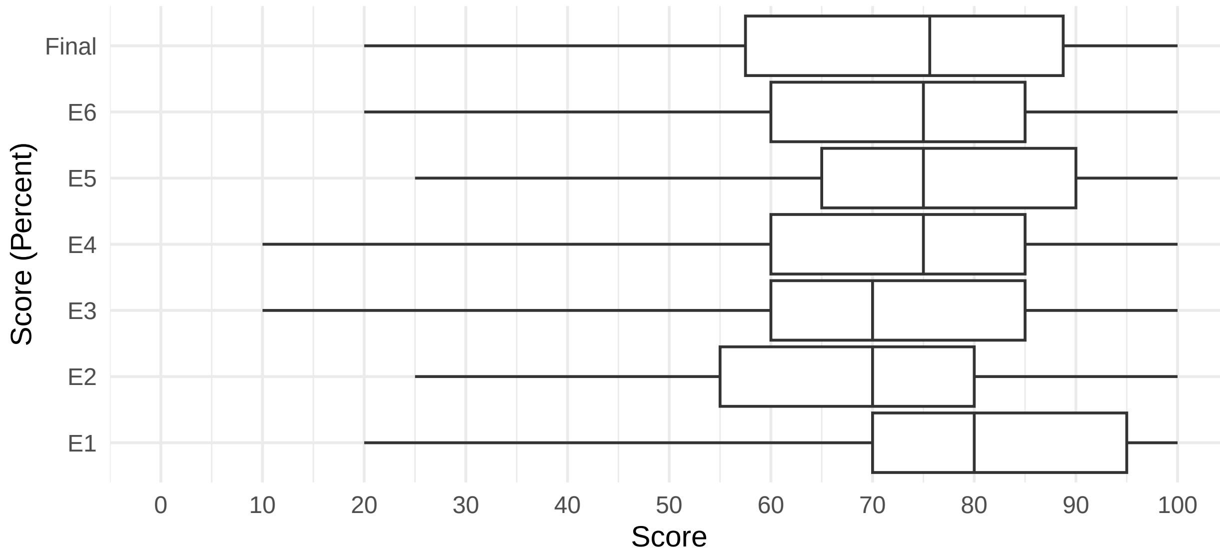
A box plot visually depicts three summary measures: the **median** (i.e., Q_2), **range** (i.e., maximum − minimum), and **interquartile range** (i.e., $Q_3 - Q_1$).

Example: Cumulative distributions and box plots of the data from the study on creativity and motivation.

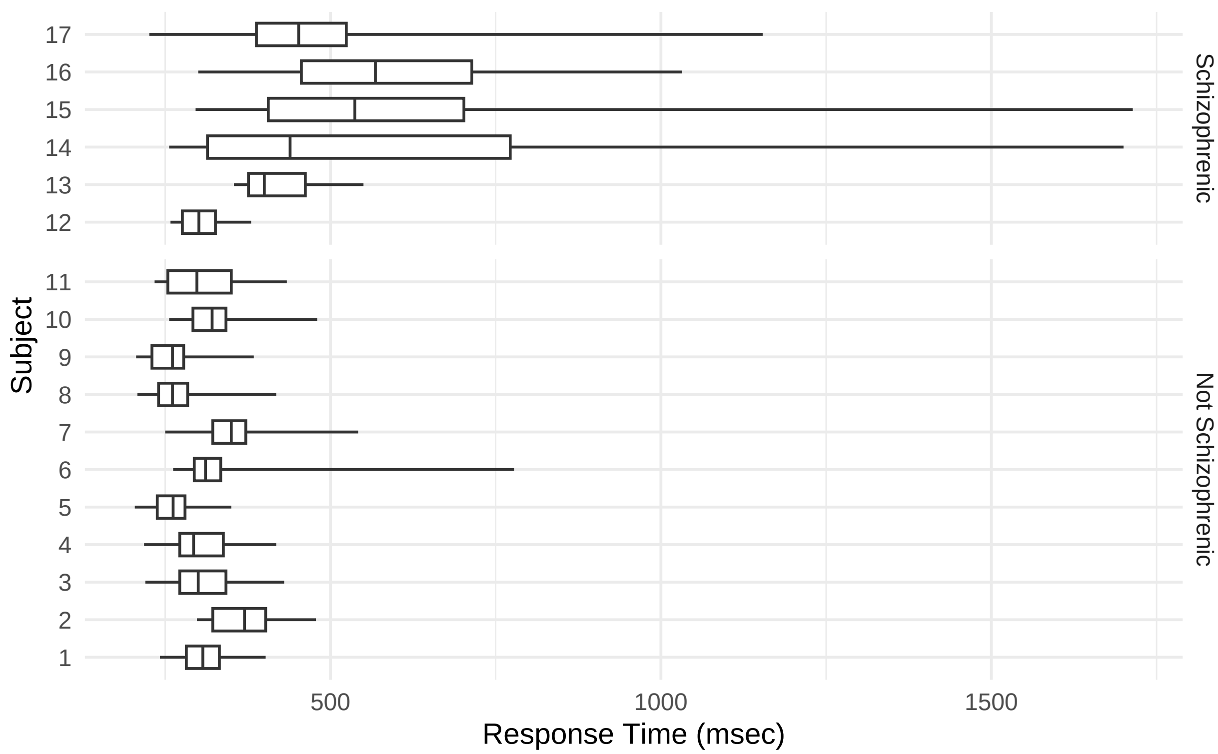




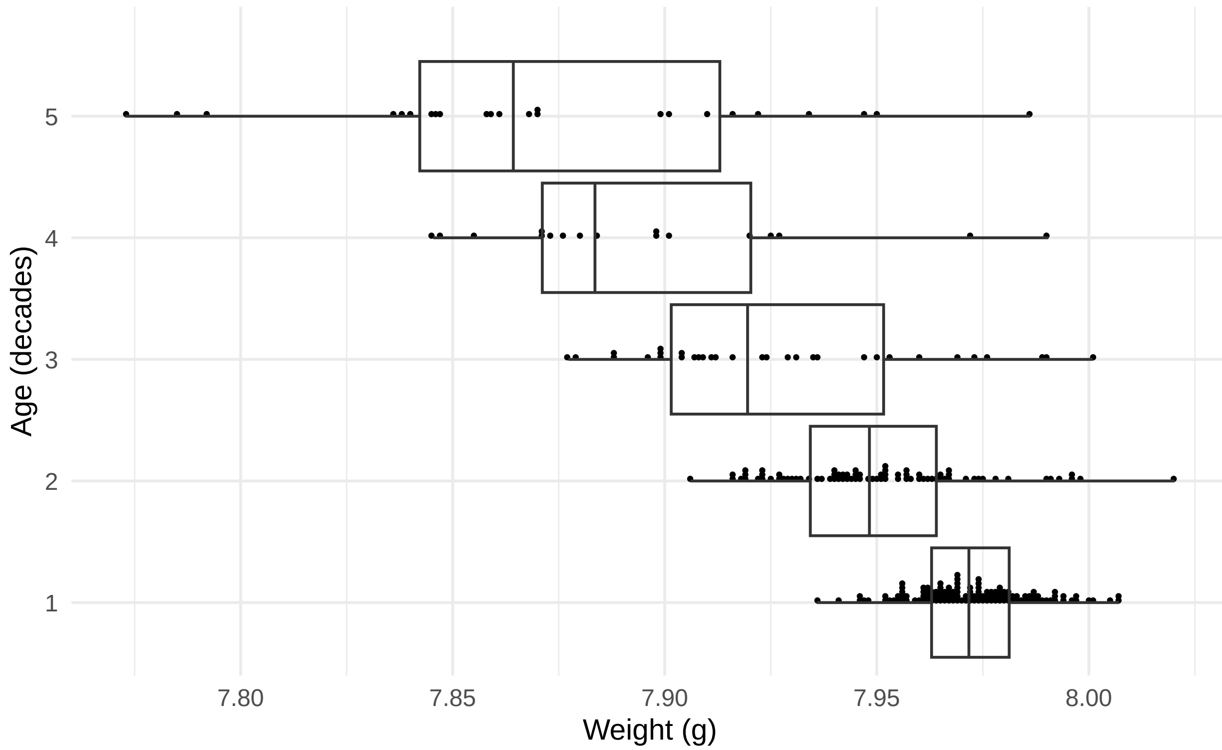
Example: Distribution of examination scores from Stat 251, Fall 2016.



Example: Distribution of response times for 11 non-schizophrenic individuals and six schizophrenic individuals.



Example: Distributions of samples of observations of the weights of gold sovereigns collected from circulation in Manchester, England.

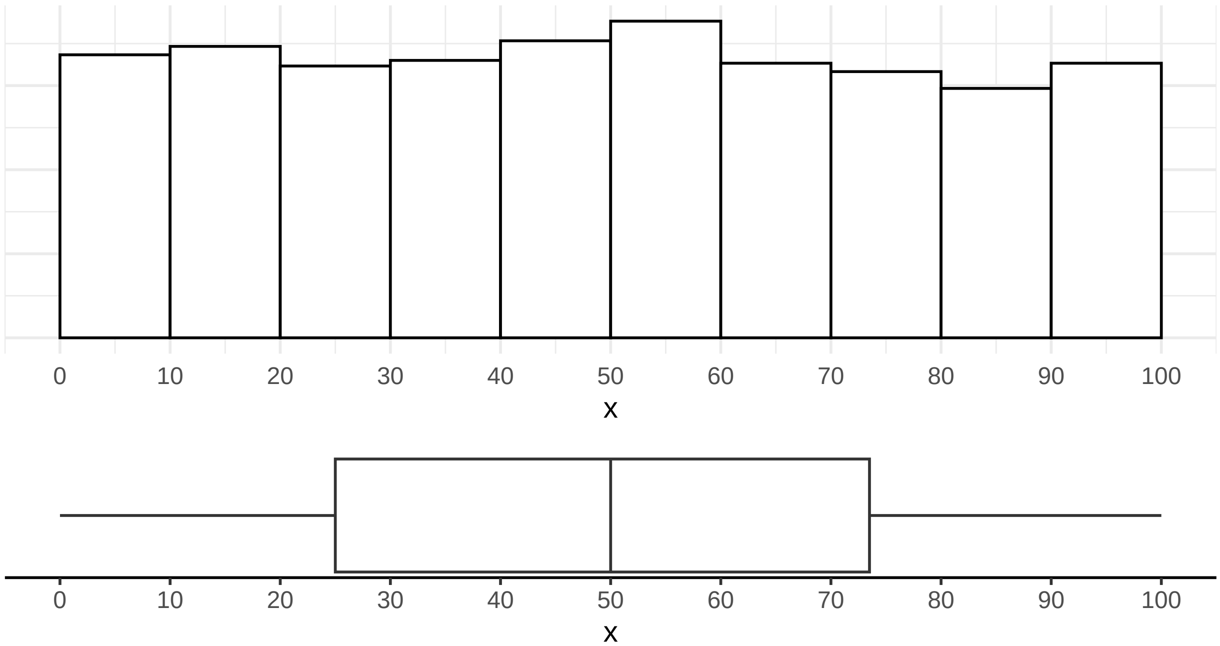


Age	n	mean	sd	min	Q1	Q2	Q3	max	IQR
5	24	7.873	0.0535	7.773	7.842	7.864	7.913	7.986	0.0708
4	17	7.896	0.0406	7.845	7.871	7.883	7.920	7.990	0.0492
3	32	7.928	0.0343	7.877	7.902	7.920	7.952	8.001	0.0501
2	78	7.950	0.0227	7.906	7.934	7.948	7.964	8.020	0.0297
1	123	7.973	0.0141	7.936	7.963	7.972	7.981	8.007	0.0183

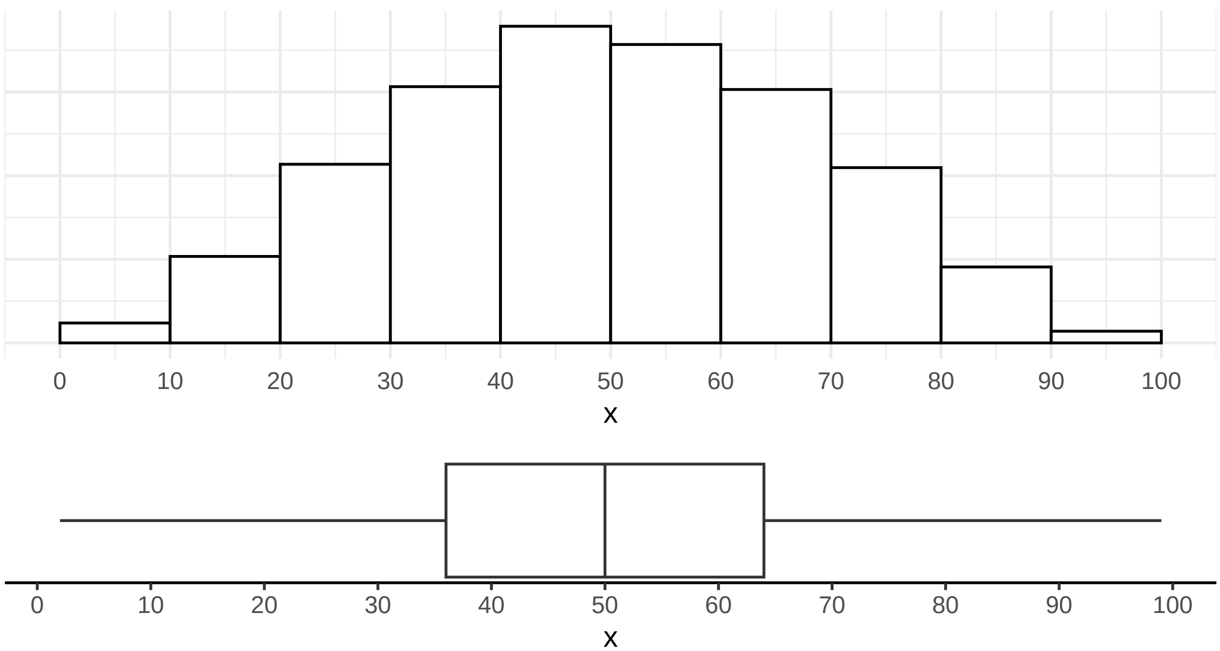
Distribution Shapes

Some terms we use to describe the *shape* of the distribution of a quantitative variable: *symmetric*, *uniform*, *left-skewed*, *right-skewed*, *uni-modal*, *bi-modal*.

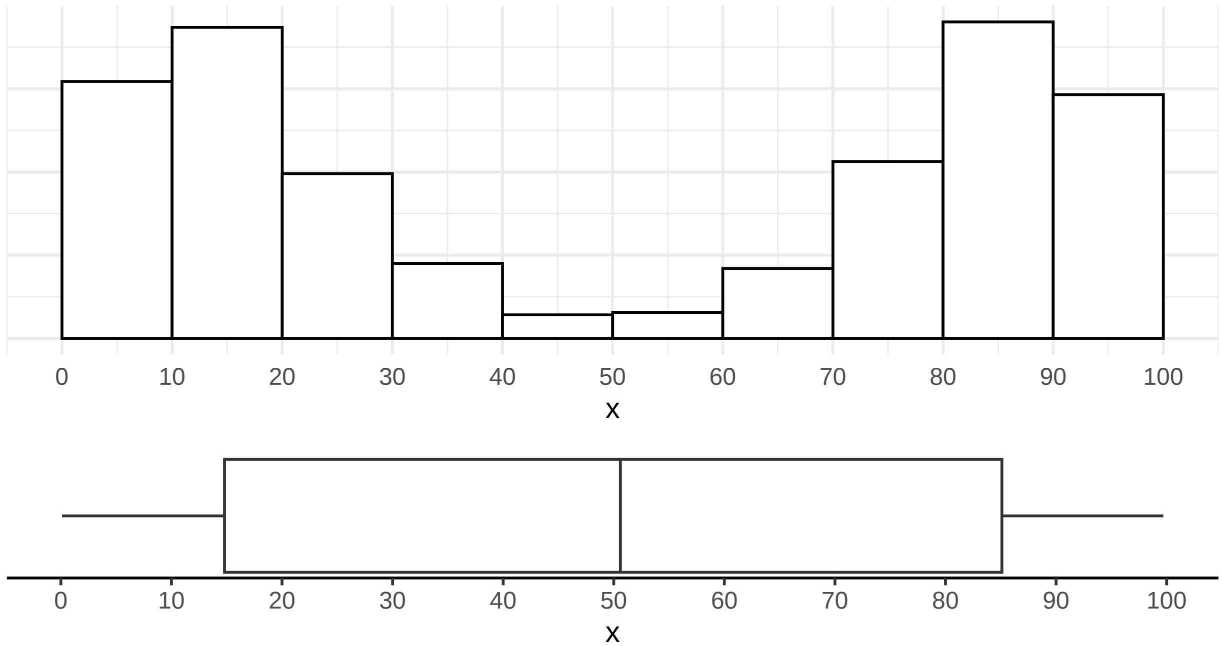
Example: Uniform and symmetric.



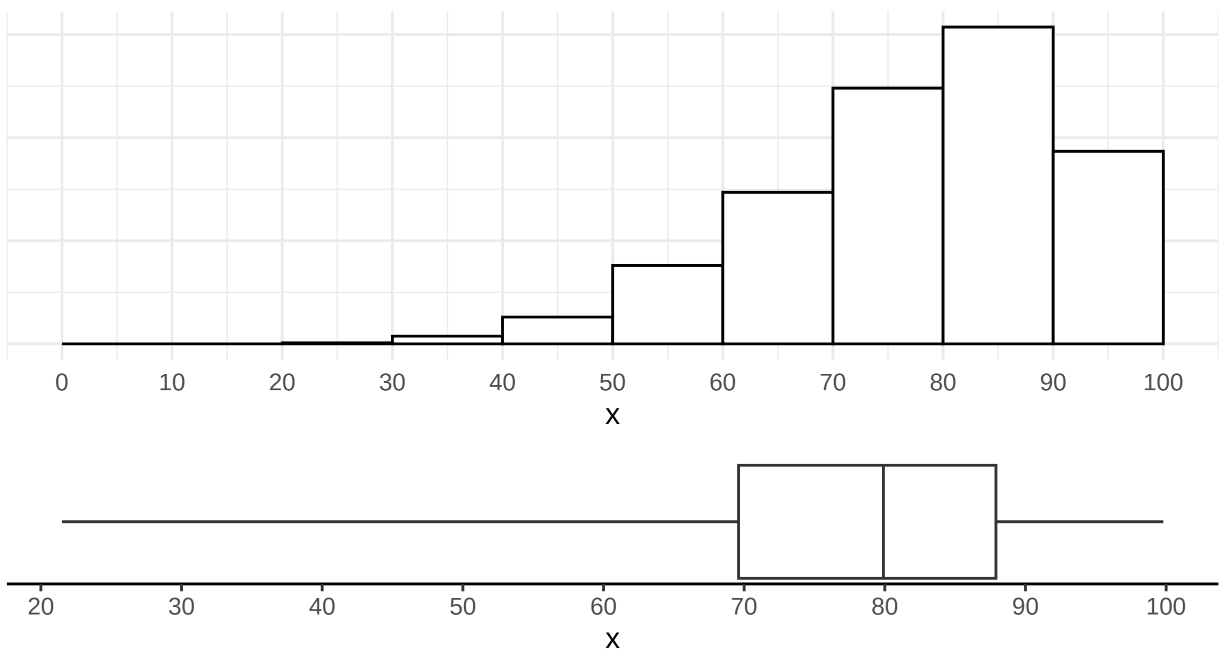
Example: Uni-modal and symmetric.



Example: Bi-modal and symmetric.



Example: Left-skewed and uni-modal.



Example: Right-skewed and uni-modal.